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# THE UNITARIZABILITY OF THE AUBERT DUAL OF STRONGLY POSITIVE SQUARE INTEGRABLE REPRESENTATIONS

BY

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ABSTRACT

Under a natural assumption, which holds in the generic case, we prove in this paper that, for the classical $p$-adic groups, the Aubert dual of an irreducible, strongly positive square-integrable representation (in the sense of Mœgl.,in-Tadić classification), is unitarizable. In this way, for this important class of representations, we verify the conjecture which states that the Aubert involution preserves unitarity.

## 1. Introduction

The problem of classification of the unitary dual of classical $p$-adic groups is very important, and largely, unsolved. The unitarizability problem is an important problem in several aspects: unitary representations are crucial in the harmonic analysis on the classical $p$-adic groups, generalizing the classical commutative theory. On the other hand, unitarizable representations occur at the local places of the automorphic representations, and, as such, have a number-theoretic significance.

This problem is completely solved only for the general linear groups (Tadić, Vogan), and for some groups of small rank: for, e.g., $GSp(4)$ and $Sp(4)$ (Tadić, Sally) and some others. Also, generic unitary dual is classified for the quasi-split classical groups (Lapid, Muić, Tadić), and there is also a work of Barbasch and

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Moy in the spherical case. As for the general linear groups over division algebras, their unitary dual is classified modulo a conjecture which states that the representation parabolically induced from an irreducible unitary representation is irreducible.

In this paper, we tackle the unitarizability problem using the Aubert (i.e., generalized Zelevinsky's involution). The involution on the Grothendieck group of the smooth, finite length representations of a reductive group was studied by many, including Zelevinsky, Iwahori, Matsumoto, Casselman, Bernstein, Barbasch, Moy, Schneider and Sthuler, but we use the definition which works for a general reductive $p$-adic group as given in [2], and refer to it as the Aubert involution.

One of the most intriguing conjectures about Aubert involution states that it preserves unitarity. This conjecture was posed by Bernstein for the general linear groups in [4] and proved by Tadić (in the case of general linear groups). Also, Barbasch and Moy proved that, in the Iwahori case, involution preserves unitarity.

It would be very helpful to know that this conjecture holds for the square-integrable representations (of the classical groups). In this paper we start to address this case by treating a special kind of the square-integrable representations, namely the strongly positive ones. They serve as the "building blocks" for all the square-integrable representations, as can be seen from the Mœglin-Tadić classification. They include generalized Steinberg representations, regular discrete series etc.

To prove the unitarizability of Aubert dual of a strongly positive square-integrable representation, we, roughly, proceed as follows: assume that $\sigma$ is an irreducible, strongly positive discrete series; we denote its Aubert dual by $\widehat{\sigma}$. We find an irreducible square-integrable (mod center) representation $\delta$ of a general linear group such that the induced representation $\delta \rtimes \sigma$ is irreducible (the notation is explained in Section 2). The first point of reducibility of the series of representations $\delta\nu^s \rtimes \sigma$, where $s$ is real and $s \geq 0$, in this particular case, is $s = 1/2$. Thanks to the results in [8] and [9], we know all the irreducible subquotients of $\delta\nu^{1/2} \rtimes \sigma$. Using an inductive argument, we can prove that Aubert duals of all these subquotients are unitarizable (because they appear as a subquotients of the similar induced representation whose unitarizability follows from a certain inductive argument). Now we calculate the signature of the hermitian forms involved, and we get that the representation $\widehat{\delta} \rtimes \widehat{\sigma}$ is unitarizable (and we know that it is irreducible and hermitian), forcing representation $\widehat{\sigma}$ to be unitarizable, too.

Our approach uses results of Muić about reducibility and the composition series of the generalized principal series [12] and [13]; some reducibility results we use can be derived form the earlier results of Tadić [20], Jantzen [6] and others.

Now we describe the content of the paper, section by section.

In Section 2, we recall the classification of the discrete series representations of classical $p$-adic groups in terms of the admissible triples [8], [9], and the notion of a strongly positive discrete series.

In the third section, we prove the unitarizability of the generalized Steinberg representation using a simple idea of analyzing the ends of the "complementary series" and proving that all the appearing subquotients are unitarizable. Also, in this section we prove several more general results describing the structure of the induced representations involved, and these results are used in the general case of strongly positive discrete series.

In the fourth section, we prove that the Aubert dual of a strongly positive discrete series whose cuspidal support on general linear group-side consists only of the twists of one irreducible, self-dual cuspidal representation, is unitarizable. We also introduce two-fold inductive procedure which we use for proving the unitarizability; the case of the generalized Steinberg representation is used as a basis of this procedure.

In the fifth section, using the same idea as in the fourth section, we prove our statement for a general strongly positive discrete series. We also prove some auxiliary statements which were self-evident in the previous cases (covered in the third and the fourth sections).

Throughout the paper we assume that the basic assumption (which follows from certain Arthur's conjectures, [8],[9]) holds. We can formulate this assumption in the following way: For an irreducible, self-dual, cuspidal representation $\rho$ of the group $GL(n, F)$ ($F$ is $p$-adic) and an irreducible, cuspidal representation $\sigma$ of the classical group, there exists a unique $\alpha_{\rho,\sigma} \geq 0$ such that $\nu^{\alpha_{\rho,\sigma}}\rho \rtimes \sigma$ reduces (this is proved in [18]; the notation is explained in Section 2). The basic assumption states that $\alpha_{\rho,\sigma} - \alpha_{\rho,1} \in \mathbb{Z}$ (here 1 denotes the trivial representation of the trivial group). F. Shahidi has proved that $\alpha_{\rho,1} \in \frac{1}{2}\mathbb{Z}$, moreover, he proved that the basic assumption holds if $\sigma$ is generic.

It is important to note that, in cases in which it is known that the basic assumption holds (for example, when the square-integrable representation we are studying is generic), our results are complete and there are no additional conditions or hypothesis.

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## 2. Notation and Preliminaries

Let $F$ be a nonarchimedean field of characteristic different from 2. Let $\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{C}$ denote the ring of rational integers, positive rational integers and the field of real numbers, the field of complex numbers, respectively.

The groups we are considering are of the following form: We have a tower of the (full) orthogonal or symplectic groups $G_n = G(V_n)$, which are the groups of isometries of $F$-vector spaces $V_n$, endowed with the non-degenerate form. The form is symmetric if the tower is orthogonal, and skew-symmetric, otherwise. The subscript "$n$" denotes the split rank of the group $G_n$.

We now review some basic facts related to the representation theory of the general linear groups [23]. By $\nu$ we denote a composition of the determinant mapping with the normalized (in a usual way) absolute value on $F$. Let $\rho$ denote an irreducible cuspidal representation of $GL(n, F)$. Then, by a segment of cuspidal representations, denoted $[\rho, \rho\nu^m]$, we mean an (ordered) set $\{\rho, \rho\nu, \rho\nu^2, \ldots, \rho\nu^m\}$. To each such segment we attach an irreducible essentially square-integrable representation, denoted $\delta([\rho, \rho\nu^m])$, which is a unique irreducible subrepresentation of $\rho\nu^m \times \rho\nu^{m-1} \times \cdots \times \rho$. Here we use a well known notation for the normalized parabolic induction for the general linear groups, with the usual choice of the standard parabolic subgroups. Let $\sigma_1, \ldots, \sigma_k$ denote square-integrable representations of the general linear groups. If the real numbers $s_1, s_2, \ldots, s_k$ satisfy $s_1 \geq s_2 \geq \cdots \geq s_k$, the representation $\sigma_1\nu^{s_1} \times \sigma_2\nu^{s_2} \times \cdots \times \sigma_k\nu^{s_k}$ has a unique irreducible quotient, the Langland's quotient, which we denote by $L(\sigma_1\nu^{s_1}, \sigma_2\nu^{s_2}, \ldots, \sigma_k\nu^{s_k})$.

Now we recall the corresponding notation for the classical groups. If $\pi_i$, $i = 1, \ldots, k$ is a representation of the group $GL(n_i, F)$ and if $\tau$ is a representation of the group $G_m$, then by $\pi_1 \times \pi_2 \times \cdots \times \pi_k \rtimes \tau$ we denote a parabolically induced representation of the group $G_n$, induced from the standard parabolic subgroup with the Levi subgroup isomorphic to $GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_m$. Here $n = n_1 + n_2 + \cdots + n_k + m$, and we make the usual choice of the minimal parabolic subgroup which consists of the upper-triangular matrices. Let $\sigma_i$, $i = 1, \ldots, k$ be a square-integrable representation of $GL(n_i, F)$. If $\tau$ is an irreducible tempered representation of the group $G_m$, $\sigma_i$'s and $s_1, s_2, \ldots, s_k$ satisfy the conditions as above, and, additionally, $s_k > 0$, then, analogously, the Langland's quotient of the representation $\sigma_1\nu^{s_1} \times \sigma_2\nu^{s_2} \times \cdots \times \sigma_k\nu^{s_k} \rtimes \tau$ is denoted by $L(\sigma_1\nu^{s_1}, \sigma_2\nu^{s_2}, \ldots, \sigma_k\nu^{s_k}; \tau)$.

For a reductive group $G$, by $R(G)$ we denote the Grothendieck group of smooth, finite length representations of $G$. Let $R = \bigoplus_{n\geq 0} R(GL(n, F))$, and $R_1 = \bigoplus_{n\geq 0} G_n$. Here $GL(0, F)$ and $G_0$ denote the trivial group. For a representation $\sigma$ of the group $G_n$, by $s_k(\sigma)$ we denote the (normalized) Jacquet module of $\sigma$ with respect to the standard Levi subgroup isomorphic to $GL(k, F) \times G_{n-k}$. For an irreducible representation $\sigma$ of some $G_n$, we introduce

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(s_k(\sigma)),$$

where "s.s." stands for the semisimplification. We extend $\mu^*$ by linearity to whole $R_1$, and we get a mapping $\mu^* : R_1 \to R \otimes R_1$. In the same way, for an irreducible representation $\pi$ of $GL(n, F)$, let $r_k(\pi)$ denote Jacquet module of $\pi$ with respect to the standard Levi subgroup isomorphic to $GL(k, F) \times GL(n-k, F)$. We introduce

$$m^*(\pi) = \sum_{k=0}^{n} \text{s.s.}(r_k(\pi)),$$

and extend $m^*$ by linearity to the mapping $m^* : R \to R \otimes R$. Denote by $\kappa : R \otimes R \to R \otimes R$ a mapping defined by $\kappa(\sum x_i \otimes y_i) = \sum y_i \otimes x_i$. We introduce a homomorphism $M^* : R \to R \otimes R$ in the following way

$$M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ \kappa \circ m^*.$$

Then, for $\pi \in R$ and $\sigma \in R_1$, the following holds (Theorems 5.4 and 6.5 of [19])

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma). \tag{1}$$

For an irreducible representation $\sigma$ of the general linear or a classical group we consider, its Aubert dual [2] (in the Grothendieck group) is also, up to a sign, an irreducible representation. This irreducible (actual) representation we denote by $\hat{\sigma}$.

Now we briefly recall the definition of the invariants of the discrete series of the groups $G_n$, namely, the definitions of the Jordan block, function $\epsilon$ and the cuspidal support [8] and [9].

A partial supercuspidal support of a discrete series representation $\sigma$ of $G_n$ is an irreducible supercuspidal representation $\sigma_{cusp}$ of some $G_{m'}$ such that there exists an irreducible admissible representation $\pi$ of some $GL(m_\pi, F)$ (this defines $m_\pi$) such that $\sigma$ is a subrepresentation of $\pi \rtimes \sigma_{cusp}$.

The set $\mathrm{Jord}(\sigma)$ is defined as a set of all pairs $(a, \rho)$ where $\rho \cong \widetilde{\rho}$ is an irreducible supercuspidal representation of some $GL(m_\rho, F)$ and $a > 0$ is an integer such that both of the following properties are satisfied:

(i) $a$ is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$. The local $L$-function $L(s, \rho, r)$ is the one defined by Shahidi (for example, [16]), where $r = \Lambda^2\mathbb{C}^{m_\rho}$ is the exterior square of the standard representation if $G_n$ is symplectic or even orthogonal group, and $r = \mathrm{Sym}^2\mathbb{C}^{m_\rho}$ if $G_n$ is odd orthogonal.

(ii) the induced representation

$$\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \sigma$$

is irreducible.

We do not recall the exact definition of the function $\epsilon$; it can be found in [8]. These invariants completely describe the representation $\sigma$.

Now we briefly recall the notion of the triple. $(\mathrm{Jord}, \sigma', \epsilon)$ is a triple if $\sigma'$ is an irreducible supercuspidal representation of some $G_m$, Jord is a finite set (possibly empty) of pairs $(a, \rho)$ which satisfy the property (i) from above. $\epsilon$ is a function, partially defined on $\mathrm{Jord} \cup \mathrm{Jord} \times \mathrm{Jord}$. (We will not recall the definition and requirements on $\epsilon$; we refer to [9] and [12].

We introduce $\mathrm{Jord}_\rho = \{a : (a, \rho) \in \mathrm{Jord}\}$. For $a \in \mathrm{Jord}_\rho$ we define (if it exists) $a_- = \max\{b \in \mathrm{Jord}_\rho : b < a\}$.

For the present purpose, the most important notion is that of an alternated triple. The triple $(\mathrm{Jord}, \sigma', \epsilon)$ is of alternated type if for any $\rho$ such that $\mathrm{Jord}_\rho \neq \emptyset$, the following holds:

1. If $a \in \mathrm{Jord}_\rho$ such that $a_-$ is defined, then

$$\epsilon(a, \rho)\epsilon(a_-, \rho) = -1.$$

2. There is an increasing bijection $\phi_\rho : \mathrm{Jord}_\rho \to \mathrm{Jord}'_\rho(\sigma')$, where

$$\mathrm{Jord}'_\rho(\sigma') = \begin{cases} \mathrm{Jord}_\rho(\sigma') \cup \{0\} & \text{if } a \text{ is even and } \epsilon(\min \mathrm{Jord}_\rho) = 1; \\ \mathrm{Jord}_\rho(\sigma') & \text{otherwise.} \end{cases}$$

The admissible triples are obtained from the alternated ones by the successive relation of dominance [9].

The classification of Mœglin and Tadić [9] states that there is one-to-one correspondence between the set of the equivalence classes of the discrete series representations (of all the groups in one tower) and the set of all admissible triples. Namely, the triple of invariants of an irreducible square-integrable representation is an admissible triple, and, vice versa, each admissible triple is triple of invariants of some square-integrable representation.

A square integrable representation is of a strongly positive type if the corresponding triple is of an alternated type.

We can explicitly characterize strongly positive discrete series as follows: If $(\mathrm{Jord}, \sigma', \epsilon)$ is an alternated triple corresponding to the discrete series $\sigma$, then $\sigma$ is a unique subrepresentation of

$$\times_\rho \times_{i=1}^{k_\rho} \delta([\nu^{\frac{\phi_\rho(a_i^\rho)+1}{2}}\rho, \nu^{\frac{a_i^\rho - 1}{2}}\rho]) \rtimes \sigma'.$$

Here, for each $\rho$ such that $\mathrm{Jord}_\rho \neq \emptyset$, we have written down the elements of $\mathrm{Jord}_\rho$ in the increasing order

$$a_1^\rho < a_2^\rho < \cdots < a_{k_\rho}^\rho.$$

We have the following important characterization of the strongly positive discrete series.

Proposition 2.1 ([12]): *Let $\sigma$ be a discrete series representation of $G_n$. Then, $\sigma$ is strongly positive if and only if for each representation*

$$\rho_1\nu^{s_1} \times \cdots \times \rho_k\nu^{s_k} \rtimes \sigma_{cusp},$$

*where $\rho_i$, $i = 1, \ldots, k$ are irreducible cuspidal representations and $s_i$, $i = 1, \ldots, k$ real numbers, such that*

$$\sigma \hookrightarrow \rho_1\nu^{s_1} \times \cdots \times \rho_k\nu^{s_k} \rtimes \sigma_{cusp},$$

*we have $s_i > 0$ for each $i$.*

## 3. Generalized Steinberg representation

Let $\rho \cong \tilde{\rho}$ be an irreducible unitary cuspidal representation of the group $GL(k,F)$, and $\sigma'$ an analogous representation of $G_m$. Assume that there exists $\alpha > 0$ such that $\rho\nu^{\alpha} \rtimes \sigma'$ reduces. (There always exists a unique $s_0 \geq 0$ such that $\rho\nu^{s_0} \rtimes \sigma'$ reduces, [18]). For any integer $n \geq 0$, the representation

$$\rho\nu^{n+\alpha} \times \rho\nu^{n-1+\alpha} \times \cdots \times \rho\nu^{\alpha} \rtimes \sigma'$$

has a unique irreducible subrepresentation [21], which we denote by $\sigma_n$. The representation $\sigma_n$ is a square-integrable representation [21], and because of the obvious analogy with the Steinberg representation for the classical groups (which is the special case of the above construction, obtained for the trivial representation $\sigma'$ of the trivial group, and the trivial character $\rho$ of $GL(1,F)$), is called a generalized Steinberg representation.

In this paper, we shall consider the case when $\alpha$ belongs to $\frac{1}{2}\mathbb{Z}$. This is expected to be always the case. Further, basic assumption would imply this.

We now describe the Jordan block and the function $\epsilon$ attached to the generalized Steinberg representation. The generalized Steinberg representation belongs to the set $\mathcal{D}(\rho,\sigma')$ [7] (the set of all the square-integrable representations of the classical groups in one tower, such that their partial cuspidal support is $\sigma'$, and the rest of the cuspidal support on the $GL$-side is formed from the twists of the representation $\rho$). Because of that, we consider the function $\epsilon$ as a function on $\mathrm{Jord}_\rho$ or $\mathrm{Jord}_\rho \times \mathrm{Jord}_\rho$.

(1) If $\alpha = 1/2$, then $\mathrm{Jord}_\rho(\sigma') = \emptyset$ (so the function $\epsilon$ is not defined on $\mathrm{Jord}_\rho(\sigma')$), and $\mathrm{Jord}_\rho(\sigma_n) = \{2n+2\}$ with $\epsilon(2n+2) = 1$.

(2) If $\alpha \in 1/2+\mathbb{Z}$, and $\alpha > 1/2$, then $\mathrm{Jord}_\rho(\sigma') = \{2,\ldots,2\alpha-3,\, 2\alpha-1\}$, with the alternating function $\epsilon$ on $\mathrm{Jord}_\rho(\sigma')$ with $\epsilon(2) = -1$. Then $\mathrm{Jord}_\rho(\sigma_n) = \{2,\ldots,2\alpha-3,\, 2n+2\alpha+1\}$, with the alternating function $\epsilon$ on $\mathrm{Jord}_\rho(\sigma_n)$ with $\epsilon(2) = -1$.

(3) If $\alpha \in \mathbb{Z}_{>0}$, then $\mathrm{Jord}_\rho(\sigma') = \{1,\ldots,2\alpha-3,\, 2\alpha-1\}$ with the alternating function $\epsilon$ defined only on pairs, and analogously, $\mathrm{Jord}_\rho(\sigma_n) = \{1,\ldots,2\alpha-3,\, 2n+2\alpha+1\}$, with the alternating function $\epsilon$ defined only on pairs.

To prove that the Aubert dual $\hat{\sigma_n}$ of the generalized Steinberg representation $\sigma_n$ is an unitarizable representation, we use the following simple idea: for a certain discrete series representation $\delta_n$ of the general linear group, such that

$\delta_n \rtimes \sigma_n$ is irreducible, we prove that the representation $\hat{\delta_n} \rtimes \hat{\sigma_n}$ (again, necessarily irreducible) is unitarizable. Because the representation $\hat{\delta_n} \otimes \hat{\sigma_n}$ is Hermitian, from this follows that the representation $\hat{\sigma_n}$ is unitarizable.

To prove the unitarizability of the representation $\hat{\delta_n} \rtimes \hat{\sigma_n}$, we find the first positive reducibility point of the representation $\nu^s \hat{\delta_n} \rtimes \hat{\sigma_n},\ s \geq 0$, then prove (by certain inductive argument) that all the irreducible subquotients appearing in the composition series of this (reducible) representation are unitarizable. To show the unitarizability of $\hat{\delta_n} \rtimes \hat{\sigma_n}$, we use Jantzen filtration and, again, certain inductive argument.

We will denote $\sigma_{-1} = \sigma'$.

For an integer $n \geq 1$, we define

$$\delta_n = \delta([\nu^{-(n+\alpha-3/2)}\rho, \nu^{(n+\alpha-3/2)}\rho]),$$
$$\delta'_n = \delta([\nu^{-(n+\alpha-1)}\rho, \nu^{(n+\alpha-1)}\rho]),$$
$$\delta''_n = \delta([\nu^{-(n+\alpha-1/2)}\rho, \nu^{(n+\alpha-1/2)}\rho]).$$

We consider the representations $\delta_n \rtimes \sigma_n$, $\delta'_n \rtimes \sigma_{n-1}$ and $\delta''_n \rtimes \sigma_{n-2}$.

The representations $\delta_n \rtimes \sigma_n$ and $\delta''_n \rtimes \sigma_{n-2}$ are irreducible, since

$$2(n + \alpha - 3/2) + 1 \quad \text{and} \quad 2(n + \alpha - 1/2) + 1$$

differ in parity from the elements in $\mathrm{Jord}_\rho(\sigma_n)$ and $\mathrm{Jord}_\rho(\sigma_{n-2})$ (by the well-known result of Tadić (Theorem 13.2 of [20]; also see [12, Theorem 2.3]). Also, if $n = 1$ and $\alpha = 1/2$, then $\mathrm{Jord}_\rho(\sigma') = \emptyset$, and the similar argument applies.

On the other hand, $2(n+\alpha-1)+1 = 2n+2\alpha-1$ is an element of $\mathrm{Jord}_\rho(\sigma_{n-1})$, so, by the the definition of $\mathrm{Jord}_\rho$, the representation $\delta'_n \rtimes \sigma_{n-1}$ is irreducible.

We now consider the representation $\nu^s \delta_n \rtimes \sigma_n,\ s \geq 0$. The first point of reducibility is $s = 1/2$, and we analyze

$$\nu^{\frac{1}{2}} \delta_n \rtimes \sigma_n = \delta([\nu^{-(n+\alpha-2)}\rho, \nu^{n+\alpha-1}\rho]) \rtimes \sigma_n. \tag{2}$$

Proposition 3.1: *Assume that the representations $\hat{\sigma_{n-1}}$ and $\hat{\sigma_{n-2}}$ are unitarizable. Then all the irreducible subquotients of the representation $\nu^{\frac{1}{2}} \hat{\delta_n} \rtimes \hat{\sigma_n}$ are unitarizable.*

*Proof.* Actually, we will prove this proposition for all cases except when both $n = 1$ and $\alpha = 1/2$ because for this case the structure of the representation $\nu^{\frac{1}{2}} \delta_n \rtimes \sigma_n$ is different from the rest of the cases.

Using the notation of [12], for each of the essentially square-integrable representations $\delta_*$ of the general linear group which we consider, we introduce numbers $l_1$ and $l_2$ such that $\delta_* = \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho])$. For an essentially square-integrable representation $\nu^{1/2}\delta_n$ from equation (2) we have $l_1 = n+\alpha-2$ and $l_2 = n+\alpha-1$. We note that $l_1 \geq 0$, unless $n=1$ and $\alpha = 1/2$. We shall treat that case later, so, for now, we assume $l_1 \geq 0$. Then $2l_1+1 = 2n+2\alpha-3$ and $2l_2+1 = 2n+2\alpha-1$ so

$$[2l_1+1, 2l_2+1] \cap \mathrm{Jord}_\rho(\sigma_n) = \emptyset.$$

Then, in the appropriate Grothendieck group, the following holds

$$\nu^{1/2}\delta_n \rtimes \sigma_n = L(\nu^{1/2}\delta_n;\, \sigma_n) + \sigma_1' + \sigma_2',$$

where $\sigma_1'$ and $\sigma_2'$ are the discrete series obtained from $\sigma_n$ by extending the admissible triple of $\sigma_n$ in a way described in [9], see also [12, Theorems 2.1 and 2.3]. Again, $\sigma_1'$ and $\sigma_2'$ belong to $\mathcal{D}(\rho, \sigma')$. To be more specific:

$$\mathrm{Jord}_\rho(\sigma_i') = \mathrm{Jord}_\rho(\sigma_n) \cup \{2n+2\alpha-3,\ 2n+2\alpha-1\}, \quad i = 1,2$$

and the $\epsilon$ function on $\mathrm{Jord}_\rho(\sigma_1')$ extends the (alternating) function $\epsilon$ on $\mathrm{Jord}_\rho(\sigma_n)$ (or $\mathrm{Jord}_\rho \times \mathrm{Jord}_\rho$) in such a way that $\epsilon(2n+2\alpha-3)\epsilon(2n+2\alpha-1)^{-1} = 1$ and $\epsilon(2\alpha-3)\epsilon(2n+2\alpha-3)^{-1} = 1$; the $\epsilon$ function on $\mathrm{Jord}_\rho(\sigma_2')$ extends the function $\epsilon$ on $\mathrm{Jord}_\rho(\sigma_n)$ in a way that $\epsilon(2n+2\alpha-3)\epsilon(2n+2\alpha-1)^{-1} = 1$ and $\epsilon(2\alpha-3)\epsilon(2n+2\alpha-3)^{-1} = -1$.

We now consider the representation $\nu^s\delta_n'' \rtimes \sigma_{n-2}$. The first non-negative point of reducibility of this representation is $s = 1/2$, and

$$\nu^{\frac{1}{2}}\delta_n'' \rtimes \sigma_{n-2} = \delta([\nu^{-(n+\alpha-1)}\rho,\ \nu^{n+\alpha}\rho]) \rtimes \sigma_{n-2}.$$

Here again $l_1 = n+\alpha-1 \geq 0$, and

$$[2l_1+1,\ 2l_2+1] \cap \mathrm{Jord}_\rho(\sigma_{n-2}) = \emptyset,$$

so

$$\delta([\nu^{-(n+\alpha-1)}\rho,\ \nu^{n+\alpha}\rho]) \rtimes \sigma_{n-2} = \sigma_1'' + \sigma_2'' + L(\delta([\nu^{-(n+\alpha-1)}\rho,\ \nu^{n+\alpha}\rho]);\ \sigma_{n-2}),$$

where

$$\mathrm{Jord}_\rho(\sigma_i'') = \mathrm{Jord}_\rho(\sigma_{n-2}) \cup \{2n+2\alpha-1,\ 2n+2\alpha+1\}, \quad i = 1,2,$$

and the function $\epsilon$ on $\mathrm{Jord}_\rho(\sigma_1'')$ extends the (alternating) function $\epsilon$ on $\mathrm{Jord}_\rho(\sigma_{n-2})$ (or $\mathrm{Jord}_\rho(\sigma_{n-2} \times \mathrm{Jord}_\rho(\sigma_{n-2}))$ in such a way that

$$\epsilon(2n+2\alpha-1)\epsilon(2n+2\alpha+1)^{-1}=1 \quad \text{and} \quad \epsilon(2n+2\alpha-3)\epsilon(2n+2\alpha-1)^{-1}=1;$$

the situation is similar with $\sigma_2''$. By the comparison of the parameters, we immediately see that $\sigma_2' = \sigma_1''$.

Now, we consider the representation $\nu^s\delta_n' \rtimes \sigma_{n-1}$ for $s \geq 0$. The first reducibility point is $s=1$, and then we have

$$\nu\delta_n' \rtimes \sigma_{n-1} = \delta([\nu^{-(n+\alpha-2)}\rho, \nu^{n+\alpha-1}\rho]) \rtimes \sigma_{n-1}.$$

We have $[2l_1+1,\ 2l_2+1] \cap \mathrm{Jord}_\rho(\sigma_{n-1}) = \{2n+2\alpha-1\}$, and $2l_1+1 \notin \mathrm{Jord}_\rho(\sigma_{n-1})$, $2l_2+1 \notin \mathrm{Jord}_\rho$, so we can now apply [12, Theorem 4.1]. By comparing the admissible triples of the discrete series appearing as subquotients, we get

$$\nu\delta_n' \rtimes \sigma_{n-1} = L(\nu\delta_n';\ \sigma_{n-1}) + L(\nu^{\frac{1}{2}}\delta_n;\sigma_n) + L(\nu^{\frac{1}{2}}\delta_n'';\sigma_{n-2}) + \sigma_1' + \sigma_2''.$$

Now, we apply the well-known properties of the Aubert involution: The first reducibility point of the representation $\nu^s\widehat{\delta_n} \rtimes \widehat{\sigma_n}$, $\nu^s\widehat{\delta_n'} \rtimes \widehat{\sigma_{n-1}}$, and $\nu^s\widehat{\delta_n''} \rtimes \widehat{\sigma_{n-2}}$ for $s \geq 0$ is $1/2$, $1$ and $1/2$, respectively, and the following holds:

$$\nu^{\frac{1}{2}}\widehat{\delta_n} \rtimes \widehat{\sigma_n} = \widehat{L(\nu^{\frac{1}{2}}\delta_n;\ \sigma_n)} + \widehat{\sigma_1'} + \widehat{\sigma_2'}, \tag{3}$$

$$\nu\widehat{\delta_n'} \rtimes \widehat{\sigma_{n-1}} = \widehat{L(\nu\delta_n';\ \sigma_{n-1})} + \widehat{L(\nu^{\frac{1}{2}}\delta_n;\sigma_n)} + \widehat{L(\nu^{\frac{1}{2}}\delta_n'';\sigma_{n-2})} + \widehat{\sigma_1'} + \widehat{\sigma_2''}, \tag{4}$$

$$\nu^{\frac{1}{2}}\widehat{\delta_n''} \rtimes \widehat{\sigma_{n-2}} = \widehat{\sigma_2'} + \widehat{\sigma_2''} + \widehat{L(\nu^{\frac{1}{2}}\delta_n'';\ \sigma_{n-2})}. \tag{5}$$

We can conclude that all the irreducible subquotients of the representation $\nu^{1/2}\widehat{\delta_n} \rtimes \widehat{\sigma_n}$ (except in the case when $n=1$ and $\alpha=1/2$) are unitarizable, provided the representations $\widehat{\sigma_{n-1}}$ and $\widehat{\sigma_{n-2}}$ are unitarizable, because these subquotients appear at the ends of the complementary series described above. ∎

In the case $n=1$ and $\alpha=1/2$ we note that we have the following (by the results of Tadić [20] or [12, Theorem 5.1]):

$$\nu^{\frac{1}{2}}\rho \rtimes \hat{\sigma_1} = \widehat{L(\nu^{\frac{1}{2}}\rho;\ \sigma_1)} + \hat{\sigma_1'}, \tag{6}$$

$$\nu\delta(\widehat{[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]}) \rtimes \hat{\sigma_0} = L(\nu\delta(\widehat{[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]});\sigma_0) + \hat{\sigma_1'}, \tag{7}$$

$$\delta(\widehat{[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]}) \rtimes \hat{\sigma}' = L(\delta(\widehat{[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}]});\ \sigma') + \hat{\sigma_1'} + \hat{\sigma_2'}. \tag{8}$$

Here $\mathrm{Jord}_\rho(\sigma_i') = \{2,\ 4\},\ i = 1,2$, and $\epsilon_{\sigma_1'}(2) = \epsilon_{\sigma_1'}(4) = 1$, and $\epsilon_{\sigma_2'}(2) = \epsilon_{\sigma_2'}(4) = -1$ .

To prove the unitarizability of the representation $\hat{\delta_n} \rtimes \hat{\sigma_n}$, we will need to calculate the Jantzen filtration of the representation $\nu^s\hat{\delta_n} \rtimes \hat{\sigma_n}$ near the point $s = 1/2$. To accomplish that, we will need to know (the zeroes or the poles of) the Plancherel measure $\mu(\pm 1/2, \hat{\delta_n} \otimes \hat{\sigma_n})$.

We will calculate this Plancherel measure in a little more general context which will be of some use later, when we treat more general situation. We introduce the following notation $\delta(a,\rho) = \delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho])$.

Lemma 3.2: *Let $\rho$ be an irreducible, self-dual, cuspidal representation of $GL(k,F)$, and let $a > 2$ be a rational integer, such that $a$ is odd if and only if $L(s,\rho,r)$ has a pole for $s = 0$. Let $\sigma$ be a discrete series representation of the group $G_m$ such that $a-1,\ a+1 \notin \mathrm{Jord}_\rho(\sigma)$. Then, the Plancherel measure $\mu(s, \widehat{\delta(a,\rho)} \otimes \widehat{\sigma})$ has a simple pole for $s = \pm\frac{1}{2}$.*

*Proof.* The Plancherel measure in this (non-tempered case) is defined using the intertwining operator $A(s, \widehat{\delta(a,\rho)} \otimes \widehat{\sigma}) : \nu^s\widehat{\delta(a,\rho)} \rtimes \widehat{\sigma} \to \nu^{-s}\widehat{\delta(a,\rho)} \rtimes \widehat{\sigma}$ in the following way (up to an immaterial factor):

$$A(-s, \widehat{\delta(a,\rho)} \otimes \widehat{\sigma})A(s, \widehat{\delta(a,\rho)} \otimes \widehat{\sigma}) = \mu(-s, \widehat{\delta(a,\rho)} \otimes \widehat{\sigma})^{-1}.$$

In the previous expression, the operators $A(s, \widehat{\delta(a,\rho)} \otimes \widehat{\sigma}),\ s \in \mathbb{R}$ are the meromorphic continuations of the integral intertwining operators which converge for $s >> 0$. For the calculation of this Plancherel measure, we use the results of Ban [3, Corollary 4.2, Lemmas 6.4 and 7.1]. Although in the statements of these lemmas and corollary there is an assumption which states that $\widehat{\sigma}$ is unitarizable, this was not actually used for obtaining the following result:

$$A(-s, \widehat{\delta(a,\rho)} \otimes \widehat{\sigma})A(s, \widehat{\delta(a,\rho)} \otimes \widehat{\sigma}) = \mu(-s, \delta(a,\rho) \otimes \sigma)^{-1}.$$

So, we actually calculate (the zeroes and the poles) of the Plancherel measure attached to a standard representation $\nu^s\delta(a,\rho) \rtimes \sigma$.

By the basic assumption, the Plancherel measure $\mu(s, \delta(a, \rho) \otimes \sigma)$ is, up to an entire invertible function, equal to ([8], [9, Section 13])

$$\prod_{(\rho', a') \in \mathrm{Jord}(\sigma)} \frac{L(1+s, \delta(a,\rho) \times \delta(a',\rho'))L(1-s, \delta(a,\rho) \times \delta(a',\rho'))}{L(s, \delta(a,\rho) \times \delta(a',\rho'))L(-s, \delta(a,\rho) \times \delta(a',\rho'))} \times \frac{L(1+2s, \delta(a,\rho), r)L(1-2s, \delta(a,\rho), r)}{L(2s, \delta(a,\rho), r)L(-2s, \delta(a,\rho), r)}.$$

Now we have [10, Lemma 2.1]

$$L(s, \delta(a,\rho) \times \delta(a',\rho')) = \prod_{j=1}^{\min\{a,a'\}} L(s + \frac{a'+a}{2} - j, \rho \times \rho').$$

Since we are interested only in $s \in \mathbb{R}$, we analyze these expressions for $\rho' \cong \rho$. Then, for $s = \pm 1/2$, the only possibility for a zero or a pole to occur in the first line of the expression for the Plancherel measure, is when $a' = a+1$ or $a = a'+1$, but since $a+1,\ a-1 \notin \mathrm{Jord}_\rho(\sigma)$, this cannot happen.

Now, a zero or a pole of the Plancherel measure can only come from the expression $\frac{L(0,\delta(a,\rho),r)}{L(-1,\delta(a,\rho),r)}$. If $r = \Lambda^2 \rho_k$, then we introduce $\overline{r} = \mathrm{Sym}^2 \rho_k$, and vice versa.

Firstly, we assume that the representation $\rho\nu^{1/2} \rtimes 1$ reduces, which means that $L(s, \rho, r)$ has a pole for $s = 0$ (and also $k$ has to be even.) Then we apply Lemma 4.2 from [11] which gives us the existence of the pole of $L(s, \delta(a,\rho), r)$ for $s = 0$, and non-existence for $s = -1$. Now we want to calculate the order of this pole. From ([17], Proposition 8.1 with $a$ odd) we have:

$$L(0, \delta(a,\rho), r) = \prod_{i=1}^{\frac{a+1}{2}} L(a+1-2i, \rho, r) \prod_{i=1}^{\frac{a-1}{2}} L(a-2i, \rho, \overline{r}).$$

The only pole in the previous expression appears for $i = (a+1)/2$ in $L(0, \rho, r)$ and it is a simple pole, so $L(s, \delta(a,\rho), r)$ has a simple pole for $s = 0$.

Secondly, we assume $\rho\nu^{s'} \rtimes 1$ reduces for $s' \in \{0, 1\}$. Then $L(0, \rho, r) \neq \infty$ (e.g. [10, Lemma 2.3]), so $L(0, \rho, \overline{r}) = \infty$. Now, we apply once more [11, Lemma 4.2], and obtain $L(0, \delta(a,\rho), r) = \infty$ (assuming $(a-1)/2 \geq 1/2$). Now, we again apply [17, Proposition 8.1], $a$ is now even:

$$L(0, \delta(a,\rho), r) = \prod_{i=1}^{a/2} L(a+2i-3, \rho, r) L(a-2i, \rho, \overline{r}).$$

The only pole appears for $i = a/2$ in $L(0, \rho, \overline{r})$, again the pole in $L(0, \delta(a, \rho), r)$ is simple.

For $s = 1/2$ we get completely analogous situation. ∎

*Remark:* In almost all situations in which we need a calculation of the Plancherel measure, the conditions will be as in the previous lemma. The situations in which $a = 1$ or $a = 2$ are the following ones:

- Let $\sigma_1$ be a discrete series with the property $\mathrm{Jord}_\rho(\sigma_1) = \emptyset$ and $\nu^{1/2}\rho \rtimes \sigma_1$ reduces. Let $\sigma$ be a unique subrepresentation of $\nu^{3/2}\rho \times \nu^{1/2}\rho \rtimes \sigma_1$. We have to calculate the Plancherel measure $\mu(1/2, \rho \otimes \widehat{\sigma})$.
- Let $\sigma_1$ be a discrete series with the property $\mathrm{Jord}_\rho(\sigma_1) = \{1\}$ (and $\mathrm{Jord}_\rho(\sigma_{1,cusp}) = \{1\}$). Let $\sigma$ be a unique subrepresentation of $\nu^2\rho \times \nu\rho \rtimes \sigma_1$. We have to calculate the Plancherel measure

$$\mu(1/2, \delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \otimes \widehat{\sigma}).$$

For both cases, we calculate the Plancherel measure using the result of Heirmann, mentioned in Lemma 3.7.

We now prove a more general lemma than we need right now, but it will be of some use later. It explains a structure of the Aubert dual of the induced representation obtained when adding two more elements in the Jordan block of a strongly positive discrete series.

Lemma 3.3: *Let $\sigma$ be a strongly positive discrete series representation of the group $G_m$. Let $\rho$ be an irreducible cuspidal self-dual representation of the group $GL(k, F)$, and let $a_- < a$ be positive integers of the same parity (they are assumed to be even if and only if $L(s, \rho, r)$ has a pole for $s = 0$). Assume that $[a_-, a] \cap \mathrm{Jord}_\rho(\sigma) = \emptyset$. Then, the representation $L(\delta([\nu^{-\frac{a_--1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]); \sigma)$ is a unique irreducible quotient of the representation $\delta([\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes \widehat{\sigma}$, i.e., a unique subrepresentation of $\widehat{\delta([\nu^{-\frac{a_--1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])} \rtimes \widehat{\sigma}$.*

*Proof.* Let $\sigma_1$ and $\sigma_2$ be the (unique) subrepresentations of

$$\delta([\nu^{-\frac{a_--1}{2}}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma.$$

We denote by $t = (a + a_-)/2$. Let $G = G_{tk+m}$, $M_\Theta = M = GL(tk, F) \times G_m$. Let $w_l$ be the longest element in the (absolute) Weyl group of $G$, and let $w_{l,\Theta}$ be the longest element in $W(M_\Theta/A_\emptyset)$. Then $w = w_l w_{l,\Theta}$ is the longest element in the set $\{w' \in W : w'(\Theta) > 0\}$, and, by the well-known properties of the Aubert involution [2], we have

$$r_{M,G}(\widehat{\pi}) = w \circ D_{w^{-1}(M)} \circ r_{w^{-1}(M),G}(\pi) \tag{9}$$

for a representation $\pi$ of the group $G$. Here, $D_{w^{-1}(M)}$ denotes the Aubert involution with respect to the corresponding group; in this case $w^{-1}(M) = M$. To simplify the notation, we denote $\delta = \delta([\nu^{-\frac{a_- -1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$. In the Grothendieck group we have

$$\widehat{\delta} \rtimes \widehat{\sigma} = \widehat{\sigma_1} + \widehat{\sigma_2} + \widehat{L(\delta;\sigma)}.$$

For any irreducible subquotient $\pi$ of the representation $\widehat{\widetilde{\delta}} \rtimes \widehat{\sigma}$ to be a quotient of that representation, it is necessary that it has $\widehat{\delta} \otimes \widehat{\sigma}$ as a subquotient in the appropriate Jacquet module (this follows from the Frobenius reciprocity). Using formula (9), we see that, in that case, a representation $\widehat{\pi}$ (a subquotient of $\delta \rtimes \sigma$) should have a subquotient $\widetilde{\delta} \otimes \sigma$ in the appropriate Jacquet module (and the representation $L(\delta;\sigma)$ has that property). We will prove that $\widetilde{\delta} \otimes \sigma$ comes with the multiplicity one in $\mu^*(\delta \rtimes \sigma)$, which will then, in turn, prove our claim.

In the Grothendieck group, by (1), we have:

$$\begin{aligned} (10)\quad & M^*(\delta) \rtimes \mu^*(\sigma) \\ &= \sum_{i=-\frac{a_- -1}{2}-1}^{\frac{a-1}{2}} \sum_{j=i}^{\frac{a-1}{2}} \Big( \delta([\nu^{-i}\rho, \nu^{\frac{a_- -1}{2}}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{\frac{a-1}{2}}\rho]) \\ &\qquad \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \Big) \rtimes \mu^*(\sigma). \end{aligned}$$

We want to see when, in the above sum, the subquotient

$$\widetilde{\delta} \otimes \sigma = \delta(\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a_- -1}{2}}\rho]) \otimes \sigma$$

appears. So, let $\pi_1 \otimes \pi_2$ be an irreducible subquotient of $\mu^*(\sigma)$. Then, we analyze the summands of the form

$$\delta([\nu^{-i}\rho, \nu^{\frac{a_- -1}{2}}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{\frac{a-1}{2}}\rho]) \times \pi_1 \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \rtimes \pi_2.$$

Firstly, we consider the case $j = i$. Then the factor $\delta([\nu^{i+1}\rho, \nu^{j}\rho])$ does not exist, so then $\pi_2 = \sigma$, which forces $\pi_1 = 1$, and we then analyze

$$\delta([\nu^{-i}\rho, \nu^{\frac{a_- -1}{2}}\rho]) \times \delta([\nu^{i+1}\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes \sigma.$$

We want $\delta(\nu^{-(a-1)/2}\rho, \nu^{(a_--1)/2}\rho])$ to appear as a subquotient in the first factor of the tensor product; so we must have $\nu^{-(a-1)/2}\rho$ in the cuspidal support of $\delta([\nu^{-i}\rho, \nu^{(a_--1)/2}\rho]) \times \delta([\nu^{i+1}\rho, \nu^{(a-1)/2}\rho])$. Assume $\nu^{-(a-1)/2}\rho$ appears in the cuspidal support of $\delta([\nu^{-i}\rho, \nu^{(a_--1)/2}\rho])$. From this we conclude that $i \geq (a-1)/2$, which forces $i = (a-1)/2$ and then the summand is actually equal to $\delta(\nu^{-(a-1)/2}\rho, \nu^{(a_--1)/2}\rho]) \otimes \sigma$. If we assume it appears in $\delta([\nu^{i+1}\rho, \nu^{(a-1)/2}\rho])$ we get a contradiction.

Assume now $j > i$. We want to have

$$\delta(\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \leq \delta([\nu^{-i}\rho, \nu^{\frac{a_--1}{2}}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{\frac{a-1}{2}}\rho]) \times \pi_1.$$

But, on the right-hand side we have $\nu^{(a-1)/2}\rho$ in the cuspidal support, and on the left-hand side we do not, unless $j+1 > (a-1)/2$ which leads to $j = (a-1)/2$. Then, we must have

$$\widetilde{\delta} = \delta([\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \leq \delta([\nu^{-i}\rho, \nu^{\frac{a_--1}{2}}\rho]) \times \pi_1.$$

Since $\widetilde{\delta}$ is non-degenerate, by the results of Zelevinsky, $\pi_1$ has to be non-degenerate, too, so equal to a product of the essentially square-integrable representations, and irreducible. This leads to $\pi_1 = \delta([\nu^{-(a-1)/2}\rho, \nu^{-i-1}\rho])$. So $\pi_1 \otimes \pi_2 = \delta([\nu^{-(a-1)/2}\rho, \nu^{-i-1}\rho]) \otimes \pi_2 \leq \mu^*(\sigma)$. Since $i < j = (a-1)/2$ and $-i-1 \leq (a_- - 1)/2$, this violates the square–integrability criterion of Casselman [5, Theorem 4.4.6] for $\sigma$, a contradiction. So the subquotient $\widetilde{\delta} \otimes \sigma$ appears only for $j = i = (a-1)/2$, and the multiplicity is indeed one. ∎

Now, combining Lemma 3.2 and Lemma 3.3, we obtain

Lemma 3.4: *Let $\sigma$ be a strongly positive discrete series representation of the group $G_m$. Let $\rho$ be an irreducible self-dual representation of the group $GL(k,F)$, and let $a_-$ and $a$ be positive integers with $a = a_- + 2$, (they are assumed to be even if and only if $L(s, \rho, r)$ has a pole for $s = 0$). Assume that $a$, $a_- \notin \mathrm{Jord}_\rho(\sigma)$. Assume that the intertwining operators $A(1/2) : \delta(\widehat{[\nu^{-\frac{a_--1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]}) \rtimes \widehat{\sigma} \to \delta(\widehat{[\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]}) \rtimes \widehat{\sigma}$ and $A(-1/2) : \delta(\widehat{[\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]}) \rtimes \widehat{\sigma} \to \delta(\widehat{[\nu^{-\frac{a_--1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]}) \rtimes \widehat{\sigma}$ are holomorphic. Then, we have $\mathrm{Ker}(A(1/2)) = L(\delta(\widehat{[\nu^{-\frac{a_--1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]}); \sigma)$ and $\mathrm{Ker}(A(-1/2)) = \pi$, where $\delta(\widehat{[\nu^{-\frac{a_--1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]}) \rtimes \widehat{\sigma}/L(\delta(\widehat{[\nu^{-\frac{a_--1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]}); \sigma) \cong \pi$.*

*Proof.* We denote $\delta = \delta([\nu^{-\frac{a_- -1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$. Then, we introduce $X' = \widehat{\widehat{\delta}} \rtimes \widehat{\sigma}$ and $X = \widehat{\delta} \rtimes \widehat{\sigma}$. The representation $X'$ has a unique quotient, namely $\widehat{L(\delta;\sigma)}$. The representations $\sigma_1$ and $\sigma_2$ are introduced in Lemma 3.3. There are subrepresentation spaces $W_1 \subset W \subset \widehat{\widehat{\delta}} \rtimes \widehat{\sigma}$ such that $W_1$ is a representation space of $\widehat{\sigma_1}$ (or $\widehat{\sigma_2}$) and $\widehat{\widehat{\delta}} \rtimes \widehat{\sigma}/W \cong \widehat{L(\delta;\sigma)}$ (by Lemma 3.3). If we assume that $\mathrm{Ker}(A(-\frac{1}{2})) = W_1$, then $\mathrm{Im}(A(-\frac{1}{2}))$ would be a subspace of $X$ whose irreducible subrepresentation is not $\widehat{L(\delta;\sigma)}$, which is impossible. By Lemma 3.2, we know that there exists an holomorphic function $h$ defined near $s = 1/2$ such that $h(\frac{1}{2}) \neq 0$ and

$$A(-s)\frac{1}{s-1/2}A(s) = h(s) \tag{11}$$

for $s \approx \frac{1}{2}$. If we assume that $\mathrm{Ker}(A(1/2)) = W_1'$, for a subspace $W_1' \subset X$ different from $\widehat{L(\delta;\sigma)}$, then $W_1'/\widehat{L(\delta;\sigma)} \cong \widehat{\sigma_1}$ (or $\widehat{\sigma_2}$). In an appropriate $K$-type we can pick a holomorphic section $f$ such that $f_{1/2} \in W_1' \setminus \widehat{L(\delta;\sigma)}$. In this $K$ type we have

$$A(s) = A(1/2) + (s-1/2)A'(1/2) + \cdots .$$

This means that $\lim_{s\to 1/2} \frac{1}{s-1/2}A(s)f = A'(1/2)f$, and this is not in $\widehat{L(\delta;\sigma)}$ (for an appropriate type), so $A(-1/2)A'(1/2)f = 0$, but according to (11) this cannot hold. So, we obtain that $\mathrm{Ker}(A(1/2)) = \widehat{L(\delta;\sigma)}$. ∎

In the course of the inductive procedure which we employ when we calculate the signatures, we will need the following result.

LEMMA 3.5: *Let $\sigma$ be a strongly positive discrete series, $\rho$ an irreducible self-dual cuspidal representation of $GL(k,F)$ such that $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$. Let $s \geq 1$ be a number such that $2s-1 \in \mathrm{Jord}_\rho(\sigma)$ and $2s+1 \notin \mathrm{Jord}_\rho(\sigma)$, so that a strongly positive discrete series $\sigma_1$ (with $\mathrm{Jord}_\rho(\sigma_1) = \mathrm{Jord}_\rho(\sigma) \cup \{2s+1\} \setminus \{2s-1\}$) is a unique subrepresentation of $\nu^s\rho \rtimes \sigma$. Then, $\widehat{\sigma_1}$ is a unique subrepresentation of $\nu^{-s}\rho \rtimes \widehat{\sigma}$.*

*Proof.* We will prove the equivalent statement: $\widehat{\sigma_1}$ is a unique quotient of the representation $\nu^s\rho \rtimes \widehat{\sigma}$. It is sufficient to prove that the multiplicity of the subquotient $\nu^s\rho \otimes \sigma$ in $\mu^*(\nu^s\rho \rtimes \sigma)$ is equal to 1. Let $\pi_1 \otimes \pi_2$ be an irreducible subquotient in $\mu^*(\sigma)$. Using the formula

$$\mu^*(\nu^s\rho \rtimes \sigma) = (1 \otimes \nu^s\rho + \nu^s\rho \otimes 1 + \nu^{-s}\rho \otimes 1) \rtimes \mu^*(\sigma),$$

we see that the only possibility for $\nu^s\rho\otimes\sigma$ to appear as a subquotient in $\mu^*(\nu^s\rho\rtimes\sigma)$, besides the obvious one (when $\pi_1\otimes\pi_2 = 1\otimes\sigma$), occurs when $\pi_1 = \nu^s\rho$, so $\nu^s\rho\otimes\pi_2 \le \mu^*(\sigma)$ with the property $\sigma \le \nu^s\rho\rtimes\pi_2$. If $\pi_2$ is a representation of the group $G_m$, let $P = MN$ denote a member of an associate class of the standard parabolic subgroups in $G_m$ with the property that $r_{P,G_m}(\pi_2)$ is cuspidal. So, if $\rho_1\nu^{s_1}\otimes\rho_2\nu^{s_2}\otimes\cdots\otimes\rho_l\nu^{s_l}\otimes\sigma_{cusp}$ is any irreducible subquotient of $r_{P,G_m}(\pi_2)$, this means $\nu^s\rho\otimes\rho_1\nu^{s_1}\otimes\rho_2\nu^{s_2}\otimes\cdots\otimes\rho_l\nu^{s_l}\otimes\sigma_{cusp} \le r_{P',G_{m'}}(\sigma)$, for an appropriate $P'$ and $G_{m'}$. But the representation $\sigma$ is strongly positive, which forces $s_1 > 0, s_2 > 0, \ldots, s_l > 0$. The Casselman criterion for square integrability is satisfied for the representation $\pi_2$, moreover, $\pi_2$ is strongly positive. In order for $\nu^s\rho\rtimes\pi_2$ to be reducible, and to have a square integrable subquotient (namely $\sigma$), with $s \ge 1$, the following must hold: $\mathrm{Jord}_\rho(\pi_2) \neq \emptyset$ and $2s-1 \in \mathrm{Jord}_\rho(\pi_2)$, $2s+1 \notin \mathrm{Jord}_\rho(\pi_2)$. But then, for the subrepresentation $\sigma$ we must have $2s+1 \in \mathrm{Jord}_\rho(\sigma)$, a contradiction. ∎

Proposition 3.6: *Assume that, for $n > 1$, the representations $\widehat{\sigma_{n-1}}$ and $\widehat{\sigma_{n-2}}$ are unitarizable. Then, if the basic assumption holds, the representation $\widehat{\sigma_n}$ is unitarizable.*

*Proof.* If we assume the unitarizability of the representations $\widehat{\sigma_{n-1}}$ and $\widehat{\sigma_{n-2}}$, by Proposition 3.1, we know that all the subquotients of $\nu^{1/2}\hat{\delta_n}\rtimes\hat{\sigma_n}$ are unitarizable. To prove the unitarizability of the representation $\hat{\delta_n}\rtimes\hat{\sigma_n}$, we will calculate the signature of the (group-invariant) Hermitian form existing on the representation space of this representation using Jantzen filtration. Now we recall the definition of the Jantzen filtration in general [22].

For any admissible representation $(\pi, X)$ of finite length of a reductive group $G$ with the maximal compact subgroup $K$, we examine

$$\sum_{\delta\in\hat{K}} m(\delta)\delta,$$

where $m(\delta)$ is a multiplicity of the irreducible representation $\delta$ in $\pi|_K$. For every $\delta\in\hat{K}$, we have $m(\delta) < \infty$. Suppose that $\pi$ is a representation endowed with the $G$-invariant Hermitian form $\langle\cdot,\cdot\rangle$ on it's representation space $X$. For any $\delta\in\hat{K}$, we fix a positive-definite Hermitian form on the space $V_\delta$ of $\delta$. Then, the finite dimensional vector space $X^\delta = \mathrm{Hom}_G(V_\delta, K)$ is endowed with a non-degenerate hermitian form; let $(p(\delta), q(\delta))$ denote its signature. The signature of $\langle\cdot,\cdot\rangle$ is given as a formal sum $(\sum_{\delta\in\hat{K}} p(\delta)\delta, \sum_{\delta\in\hat{K}} q(\delta)\delta)$, and, for every $\delta\in\hat{K}$

, we have $m(\delta) = p(\delta) + q(\delta)$. Usually we have the following situation: we have a continuous family of the Hermitian forms (indexed by an interval) on the compact picture $X$ of the representation; for example, $\pi_s = \nu^s \pi_1 \rtimes \pi_2$, where $\pi_1$ and $\pi_2$ are irreducible and Hermitian, and $s \in [0, 1]$. Then, there exists a family of Hermitian forms on the common compact picture $X = \nu^s \pi_1 \rtimes \pi_2|_K$, induced by the intertwining operators. Vogan has shown (Theorem 3.2 and Proposition 3.3 of [22]) that the family of the Hermitian forms does not change its signature over the intervals where the representations $\pi_s$ are irreducible and that the Jantzen filtration at the reducibility point $s = s_0$ governs the signature of the Hermitian forms left ($s < s_0$) and right ($s > s_0$) from it.

In our case, the Jantzen filtration are introduced as follows: We denote by $X$ a compact picture of the representation $\hat{\delta_n} \rtimes \hat{\sigma_n}$. Then, we view the intertwining operators $A(s) = A(s, \hat{\delta_n} \otimes \hat{\sigma_n})$ as the operators on the space $X$. They induce Hermitian forms $\langle \cdot, \cdot \rangle_s$ on $X$ in the usual way, i.e.

$$\langle f_1, f_2 \rangle_s = \int_K \langle A(s) f_{1,s}(k), f_{2,s}(k) \rangle \mathrm{d}k.$$

Here $f_{1,s}$ and $f_{2,s}$ denote the holomorphic sections corresponding to $f_1, f_2 \in X$. For $s \in [0, 1/2)$ the operators $A(s)$ are isomorphisms, and the kernel of the operator $A(1/2)$ is the subspace of $X$ formed by vectors whose holomorphic sections for $s = 1/2$ form $L(\widehat{\nu^{1/2}\delta_n; \sigma_n})$. (The question of the holomorphy of this operator will be resolved in the next proposition; for now, we assume that it is holomorphic). The Jantzen filtration is a sequence [22] of $G_{m'}$-invariant spaces (where $\hat{\delta_n} \rtimes \hat{\sigma_n}$ is a representation of $G_{m'}$) given by

$$X^0_{1/2} = X \supset X^1_{1/2} = L(\widehat{\nu^{1/2}\delta_n; \sigma_n}) \supset X^2_{1/2} \supset \cdots \supset \{0\}.$$

The space $X^i_{1/2}$ is given as the radical of the hermitian form $(\cdot, \cdot)^{i-1}_{1/2}$ defined on $X^{i-1}_{1/2}$ and given by

$$\lim_{s \to 1/2} \frac{1}{(s - 1/2)^{i-1}} (A(s) \cdot, \cdot).$$

Because of the result in 3.2 (for which we needed the basic assumption), the following holds:

$$A(-s)A(s) = (s - 1/2)h(s), \tag{12}$$

where $h(s)$ is a holomorphic function near $s = 1/2$, and $h(1/2) \neq 0$.

First, we want to show that $X^2_{1/2}$ is zero, i.e. that the hermitian form $(\cdot, \cdot)^1_{1/2}$ defined on $L(\widehat{\nu^{1/2}\delta_n; \sigma_n})$ is non-degenerate. But this follows from (12); namely, we can always focus our attention on a certain $K$-type $m_\delta V_\delta$ (so $\delta$ is an irreducible representation of $K$ on the space $V_\delta$ and $m_\delta$ is the multiplicity of that representation in $X$) such that $m_\delta V_\delta \cap L(\widehat{\nu^{1/2}\delta_n;\sigma_n}) \neq \{0\}$. Then, on this $K$-type (which is a finite-dimensional subspace) we have the following expansion:

$$A(s) = A(1/2) + (s-1/2)A'(1/2) + \cdots .$$

For $f \in m_\delta V_\delta \cap L(\widehat{\nu^{1/2}\delta_n;\sigma_n})$ we have $\lim_{s\to 1/2} \frac{1}{s-1/2} A(s)f = A'(1/2)f$, so $A(-1/2)A'(1/2)f = h(1/2)f \neq 0$. Then, because $A'(1/2)f \notin \mathrm{Ker}A(-1/2)$ we can choose appropriate $v' \in L(\widehat{\nu^{1/2}\delta_n;\sigma_n})|_K$ such that $(A'(1/2)f, v') = (f,v')^1_{1/2} \neq 0$.

We denote the signature of the form $(\cdot,\cdot)^0_{1/2}$ on the quotient $X^0_{1/2}/X^1_{1/2} \cong \widehat{\sigma_1'} \oplus \widehat{\sigma_2'}$ by $(p_0,q_0)$ (this is actually a formal sum), and the signature of the form $(\cdot,\cdot)^1_{1/2}$ on $X^1_{1/2} \cong L(\widehat{\nu^{1/2}\delta_n;\sigma_n})$ by $(0,q_1)$ ($L(\widehat{\nu^{1/2}\delta_n;\sigma_n})$ is unitarizable by Proposition 3.1). The signature of the representation $\nu^s\hat{\delta_n} \rtimes \hat{\sigma_n},\ s \in [0,1/2)$ is given by $(p_0+q_1,q_0)$ (of [22, Theorem 3.2 and Proposition 3.3.]). We have to prove that $q_0 = 0$, i.e., that the form $(\cdot,\cdot)^0_{1/2}$ on $\hat{\sigma_1'} \oplus \hat{\sigma_2'}$ is definite.

We now employ an inductive procedure to calculate this signature. To emphasize the rank of the groups we are considering, we now denote $A(s) = A(s,\hat{\delta_n} \otimes \hat{\sigma_n})$ by $A(s,n)$.

By Lemma 3.5 the following holds:

$$\nu^s\hat{\delta_n} \rtimes \hat{\sigma_n} \hookrightarrow \nu^{s-(n+\alpha-3/2)}\rho \times \nu^s\widehat{\delta_{n-1}} \times \nu^{s+(n+\alpha-3/2)}\rho \times \nu^{-(n+\alpha)}\rho \rtimes \widehat{\sigma_{n-1}}.$$

Let $M_\Theta \cong GL(\overline{k},F) \times GL((2n+2\alpha-4)\overline{k},F) \times GL(\overline{k},F) \times GL(\overline{k},F) \times G_{n\overline{k}+m}$ (so that $\rho$ is a representation of the group $GL(\overline{k},F)$). Let $w$ be an element of the Weyl group $W_\Theta$ whose action on $M_\Theta$ we can describe as follows: in the usual matrix realization of the groups $G_n$ [19], $w$ transforms the block-diagonal matrix

$$diag(a,b,c,d,e,Jd^{-t}J,Jc^{-t}J,Jb^{-t}J,Ja^{-t})$$

to a matrix

$$diag(Jc^{-t}J,Jb^{-t}J,Ja^{-t}J,d,e,Jd^{-t}J,a,b,c).$$

Here $J$ denotes the matrix which generates symmetric or skew-symmetric form on the appropriate spaces, and $*^{-t}$ denotes an inverse of the matrix transpose.

We examine an intertwining operator

$$A_1(s,w) : \nu^{s-(n+\alpha-\frac{3}{2})}\rho \times \nu^s\widehat{\delta_{n-1}} \times \nu^{s+(n+\alpha-\frac{3}{2})}\rho \times \nu^{-(n+\alpha)}\rho \rtimes \widehat{\sigma_{n-1}} \to \nu^{-s-(n+\alpha-\frac{3}{2})}\rho \times \nu^{-s}\widehat{\delta_{n-1}} \times \nu^{-s+(n+\alpha-\frac{3}{2})}\rho \times \nu^{-(n+\alpha)}\rho \rtimes \widehat{\sigma_{n-1}}.$$

The integral formulas and analytic continuation imply

$$A_1(s,w)|_{\nu^s\hat{\delta_n}\rtimes\hat{\sigma_n}} = A(s,n).$$

Using the factorization of the intertwining operators [15] we get:

$$A_1(s,w) = B_1(s)A'(s,n-1)B_2(s),$$

for the intertwining operators $B_1(s)$, $A'(s,n-1)$ and $B_2(s)$. We note their actions:

$$B_2(s) : \nu^{s-(n+\alpha-\frac{3}{2})}\rho \times \nu^s\widehat{\delta_{n-1}} \times \nu^{s+(n+\alpha-\frac{3}{2})}\rho \times \nu^{-(n+\alpha)}\rho \rtimes \widehat{\sigma_{n-1}} \to \nu^{-s-(n+\alpha+\frac{3}{2})}\rho \times \nu^{s-(n+\alpha+\frac{3}{2})}\rho \times \nu^{-(n+\alpha)}\rho \times \nu^s\widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}},$$

$$A'(s,n-1) : \nu^{-s-(n+\alpha+\frac{3}{2})}\rho \times \nu^{s-(n+\alpha+\frac{3}{2})}\rho \times \nu^{-(n+\alpha)}\rho \times \nu^s\widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}} \to \nu^{-s-(n+\alpha+\frac{3}{2})}\rho \times \nu^{s-(n+\alpha+\frac{3}{2})}\rho \times \nu^{-(n+\alpha)}\rho \times \nu^{-s}\widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}},$$

$$B_1(s) : \nu^{-s-(n+\alpha+\frac{3}{2})}\rho \times \nu^{s-(n+\alpha+\frac{3}{2})}\rho \times \nu^{-(n+\alpha)}\rho \times \nu^{-s}\widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}} \to \nu^{-s-(n+\alpha-\frac{3}{2})}\rho \times \nu^{-s}\widehat{\delta_{n-1}} \times \nu^{-s+(n+\alpha-\frac{3}{2})}\rho \times \nu^{-(n+\alpha)}\rho \rtimes \widehat{\sigma_{n-1}}.$$

Let $f \in \nu^{1/2}\hat{\delta_n} \rtimes \hat{\sigma_n}$, and $f \notin L(\widehat{\nu^{1/2}\delta_n;\sigma_n})$. We want to calculate the signature of the form

$$(A_1(1/2,w)f,\ f) = (B_1(1/2)A'(1/2,n-1)B_2(1/2)f,\ f) \tag{13}$$

for $f \in \nu^{\frac{1}{2}}\hat{\delta_n} \rtimes \hat{\sigma_n} \setminus L(\widehat{\nu^{\frac{1}{2}}\delta_n;\sigma_n})$. This form is not the usual $G_{(3n+2\alpha-1)\overline{k}+m}$-invariant hermitian form on the whole space of the representation $\nu^{s-(n+\alpha-\frac{3}{2})}\rho\times \nu^s\widehat{\delta_{n-1}} \times \nu^{s+(n+\alpha-\frac{3}{2})}\rho \times \nu^{-(n+\alpha)}\rho \rtimes \widehat{\sigma_{n-1}}$; that from is induced using the long intertwining operator attached to the subset $\Theta$ of the set of all the simple roots $\Delta$. This intertwining operator $(A_1(s,w))$ induces $K$-invariant form on $\nu^{s-(n+\alpha-\frac{3}{2})}\rho \times \nu^s\widehat{\delta_{n-1}} \times \nu^{s+(n+\alpha-\frac{3}{2})}\rho \times \nu^{-(n+\alpha)}\rho \rtimes \widehat{\sigma_{n-1}}$ and $G_{(3n+2\alpha-1)\overline{k}+m}$-invariant form on $\nu^s\hat{\delta_n} \rtimes \hat{\sigma_n}$. We now introduce the compact pictures of the induced representations appearing (with the corresponding subsets $\Theta_1$ and $\Theta_2$ of simple roots): $X_\Theta = \rho \times \widehat{\delta_{n-1}} \times \rho \times \rho \rtimes \widehat{\sigma_{n-1}}$, $X_{\Theta_1} = \rho \times \rho \times \widehat{\delta_{n-1}} \times \rho \rtimes \widehat{\sigma_{n-1}}$

and $X_{\Theta_2} = \rho \times \rho \times \rho \times \widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}}$. Using the formulas for the adjoint of an intertwining operator acting on the compact picture of the representation (e.g. , [1, p. 26] or [15, Proposition 2.4.1.]) we see that

$$B_1^*(s_1, s_2, s_3, s_4; \rho\otimes\widehat{\delta_{n-1}}\otimes\rho\otimes\rho\otimes\widehat{\sigma_{n-1}}, w_2) : \nu^{s_1}\rho\times\nu^{s_2}\widehat{\delta_{n-1}}\times\nu^{s_3}\rho\times\nu^{s_4}\rho\rtimes\widehat{\sigma_{n-1}} \\ \to \nu^{s_1}\rho \times \nu^{-s_3}\rho \times \nu^{s_4}\rho \times \nu^{s_2}\widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}}.$$

Here, $w_2$ denotes the corresponding element of the Weyl group; $w_2(\Theta) = \Theta_2$. So, the equation (13) now becomes

$$(A_1(1/2, w)f,\ f) = (A'(1/2, n-1)B_2(1/2)f,\ B_1^*(1/2)f). \tag{14}$$

We want to relate the form from (14) to the form

$$(A'(1/2, n-1)B_2(1/2)f,\ B_2(1/2)f),$$

because we want to use the induction hypothesis. Using the intertwining operator

$$A(s_1, s_2, s_3, s_4; \rho \otimes \widehat{\delta_{n-1}} \otimes \rho \otimes \rho \otimes \widehat{\sigma_{n-1}}, w_1) : \\ \nu^{s_1}\rho \times \nu^{s_2}\widehat{\delta_{n-1}} \times \nu^{s_3}\rho \otimes \nu^{s_4}\rho \rtimes \widehat{\sigma_{n-1}} \to \\ \nu^{-s_3}\rho \times \nu^{s_1}\rho \times \nu^{s_4}\rho \times \nu^{s_2}\widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}},$$

where also $w_1(\Theta) = \Theta_2$, we can write

$$B_2(s) = A(s - (n + \alpha - 3/2), s, s + (n + \alpha - 3/2), -(n + \alpha); \\ \rho \otimes \widehat{\delta_{n-1}} \otimes \rho \otimes \rho \otimes \widehat{\sigma_{n-1}}, w_1).$$

By comparing the actions of the intertwining operators $B_1^*(s)$ and $B_2(s)$ on the representation

$$\nu^{s-(n+\alpha-3/2)}\rho \times \nu^{s}\widehat{\delta_{n-1}} \times \nu^{s+(n+\alpha-3/2)}\rho \times \nu^{-(n+\alpha)}\rho \rtimes \widehat{\sigma_{n-1}},$$

we see that

$$B_1^*(s)f = c(s)D(s)B_2(s)f, \tag{15}$$

where the operator $D(s)$ is the intertwining operator $A(-s - (n + \alpha - 3/2), s-(n+\alpha-3/2), -(n+\alpha), s; \rho\otimes\rho\otimes\rho\otimes\widehat{\delta_{n-1}}\otimes\widehat{\sigma_{n-1}}, w_2w_1^{-1})$, and $c(s)$ is a function, independent of $f$, which occurs as a consequence of the fact that our operators are unnormalized. The operator $D(s)$ is induced from the intertwining operator acting on $GL(2\overline{k}, F)$ which intertwines $\nu^{-s-(n+\alpha-3/2)}\rho \times \nu^{s-(n+\alpha-3/2)}\rho$ and

$\nu^{s-(n+\alpha-3/2)}\rho\times\nu^{-s-(n+\alpha-3/2)}\rho$, or, essentially $\nu^{-s}\rho\times\nu^{s}\rho$ and $\nu^{s}\rho\times\nu^{-s}\rho$, and, as such, is holomorphic for $s=\pm1/2$. Denote by $\Theta_2'$ a subset of the set of simple roots ($\Theta_2'\supset\Theta_2$) such that $M_{\Theta_2'}\cong GL(2\overline{k},F)\times GL(\overline{k},F)\times G_{(3n+2\alpha-4)\overline{k}+m}$. Then, it is well-known that

$$i_{G_{(3n+2\alpha-1)k+m},M_{\Theta_2'}}i_{M_{\Theta_2'},M_{\Theta_2}} \times(\nu^{-s-(n+\alpha-3/2)}\rho\otimes\nu^{s-(n+\alpha-3/2)}\rho\otimes\nu^{-(n+\alpha)}\rho\otimes\nu^{s}\widehat{\delta_{n-1}}\otimes\widehat{\sigma_{n-1}})\cong$$
$$i_{G_{(3n+2\alpha-1)k+m},M_{\Theta_2}} \times(\nu^{-s-(n+\alpha-3/2)}\rho\otimes\nu^{s-(n+\alpha-3/2)}\rho\otimes\nu^{-(n+\alpha)}\rho\otimes\nu^{s}\widehat{\delta_{n-1}}\otimes\widehat{\sigma_{n-1}}),$$

the isomorphisms being as follows:

$$F'\mapsto F,$$
$$F'(g)(m)=\delta_{P_{\Theta_2'}}^{-\frac{1}{2}}(m)F(mg),$$
$$F\mapsto F',$$
$$F(g)=F'(g)(1),$$

for $g\in G_{(3n+2\alpha-1)\overline{k}+m}$, $m\in M_{\Theta_2'}$. We denote by $F=B_2(1/2)f$, where $f\in\widehat{\sigma_1'}\oplus\widehat{\sigma_2'}$. Then, the following holds: for $k\in K$

$$((A'(1/2,n-1)F)(k),c(1/2)(D(1/2)F)(k))= \tag{16}$$
$$(A''(1/2,n-1)(F'(k))(1),c(1/2)D''(1/2)(F'(k))(1)). \tag{17}$$

Here, $A''(\frac{1}{2},n-1)=\mathrm{id}\otimes\mathrm{id}\otimes A(\frac{1}{2},n-1)$ is an intertwining operator acting on the space of the representation $i_{M_{\Theta_2'},M_{\Theta_2}}(\nu^{-s-(n+\alpha-3/2)}\rho\otimes\nu^{s-(n+\alpha-3/2)}\rho\otimes\nu^{-(n+\alpha)}\rho\otimes\nu^{1/2}\widehat{\delta_{n-1}}\otimes\widehat{\sigma_{n-1}})$; so the operator $A(1/2,n-1)$ acts on the representation space of the representation $\nu^{1/2}\widehat{\delta_{n-1}}\rtimes\widehat{\sigma_{n-1}}$ (as the notation suggests). The operator $c(1/2)D''(1/2)$ acts analogously, namely, the operator $D''(1/2)$ acts on the space of the representation $\nu^{-s-(n+\alpha-3/2)}\rho\times\nu^{s-(n+\alpha-3/2)}\rho$.

Let $F(k)=v_1^{(k)}\otimes v_2^{(k)}\otimes v_3^{(k)}\otimes v_4^{(k)}\otimes v_5^{(k)}$, so $v_1^{(k)}\in\nu^{-1/2-(n+\alpha-3/2)}\rho$, $v_2^{(k)}\in\nu^{1/2-(n+\alpha-3/2)}\rho$, $v_3^{(k)}\in\nu^{-(n+\alpha)}\rho$, $v_4^{(k)}\in\nu^{1/2}\widehat{\delta_{n-1}}$, $v_5^{(k)}\in\widehat{\sigma_{n-1}}$ (actually, we have a sum of the expressions of this type). The dependence on $k\in K$ is emphasized by the superscripts. At this point, we abuse the notation by identifying the representation in question with its representation space. Then, for $m_1\in GL(2\overline{k},F)$, $m_2\in GL(\overline{k},F)$, $m_3\in G_{(3n+2\alpha-4)\overline{k}+m}$, we have

$$F'(k)(m)=F'(k)(m_1,m_2,m_3)=f_1^{(k)}(m_1)\otimes(\nu^{-(n+\alpha)}\rho)(m_2)v_3^{(k)}\otimes g_4^{(k)}(m_3),$$

where $f_1^{(k)}$ and $g_4^{(k)}$ are the functions in the corresponding spaces of the induced representations with $f_1^{(k)}(1)=v_1^{(k)}\otimes v_2^{(k)}$ and $g_4^{(k)}(1)=v_4^{(k)}\otimes v_5^{(k)}$. Taking this into account, the expression given by (16) we now denote by $L(k)$ and it now reads:

$$L(k)=(f_1^{(k)}(1),c(1/2)D''(1/2)f_1^{(k)}(1))(v_3^{(k)},v_3^{(k)})(A(1/2,n-1)g_4^{(k)}(1),g_4^{(k)}(1)).$$

Let $K_{max,1}$ be a maximal compact subgroup of $GL(2\overline{k},F)$ (considered as a subgroup of $M_{\Theta_2'}$), $K_{max,2}$ a maximal compact subgroup in $G_{(3n+2\alpha-4)\overline{k}+m}$. We consider both of them as subgroups of $K$. We want to calculate the expression $L=\int_K L(k)\mathrm{d}k$, which gives the form we are interested in. Fix $k'\in K_{max,1}$. Now, we change the integration variable in the previous integral by introducing $k_1=k'k$. Since $k'\in GL(2\overline{k},F)\times\{e\}\times\{e\}\leq M_{\Theta_2'}$ , $(v_3^{(k'^{-1}k_1)},v_3^{(k'^{-1}k_1)})=(v_3^{(k_1)},v_3^{(k_1)})$ and $g_4^{(k'^{-1}k_1)}(1)=g_4^{(k_1)}(1)$. Since $L$ does not depend on $k'$, when we integrate $L$ over $K_{max,1}$, and then change the order of the integration against $K$ and $K_{max,1}$, we obtain

$$\begin{aligned}\mathrm{meas}(K_{max,1})L=&\int_K(v_3^{(k_1)},v_3^{(k_1)})(A(\frac{1}{2},n-1)g_4^{(k_1)}(1),g_4^{(k_1)}(1))\\&\times\int_{K_{max,1}}c(\frac{1}{2})(D''(\frac{1}{2})f_1^{(k_1)}(k'),f_1^{(k_1)}(k'))\mathrm{d}k'\mathrm{d}k_1.\end{aligned}$$

We denote $A(k_1)=\int_{K_{max,1}}c(\frac{1}{2})(D''(\frac{1}{2})f_1^{(k_1)}(k'),f_1^{(k_1)}(k'))\mathrm{d}k'$. Note that $A(k_1)>0$, for any $k_1\in K$. This is just a consequence of the fact that $c(\frac{1}{2})D(\frac{1}{2})(\widehat{\sigma_1'}\oplus\widehat{\sigma_2'})\neq 0$. We now have

$$\mathrm{meas}(K_{max,1})L=\int_K(v_3^{(k_1)},v_3^{(k_1)})(A(\frac{1}{2},n-1)g_4^{(k_1)}(1),g_4^{(k_1)}(1))A(k_1)\mathrm{d}k_1.$$

Now, if we change the variable of the integration again, now by fixing $k''\in\{e\}\times\{e\}\times K_{max,2}$ (so $k_2=k''k_1$), and then integrate against $K_{max,2}$, we obtain

$$\begin{aligned}\mathrm{meas}(K_{max,2})\mathrm{meas}(K_{max,1})L=&\int_K(v_3^{(k''^{-1}k_2)},v_3^{(k''^{-1}k_2)})A(k''^{-1}k_2)\\&\times\int_{K_{max,2}}(A(\frac{1}{2},n-1)g_4^{(k_2)}(k''^{-1}),g_4^{(k_2)}(k''^{-1}))\mathrm{d}k''\mathrm{d}k_2.\end{aligned}$$

We note that $(v_3^{(k''^{-1}k_2)},v_3^{(k''^{-1}k_2)})=(v_3^{(k_2)},v_3^{(k_2)})$ is a positive-definite, $K\cap GL(\overline{k},F)$-invariant form on the space of the representation $\nu^{-(n+\alpha)}\rho$.

We denote by $C(k_2) = \int_{K_{max,2}} (A(\frac{1}{2}, n-1) g_4^{(k_2)}(k''^{-1}), g_4^{(k_2)}(k''^{-1}))\mathrm{d}k''$. Note that $C(k_2) > 0$, by the induction hypothesis.

Namely, by the induction hypothesis, the representation $\widehat{\sigma_{n-1}}$ is unitarizable and for $n-1 > 1$ the structure of the representation $\nu^{\frac{1}{2}}\widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}}$ is given by Proposition 3.1. Since the kernel of the operator $A(1/2, n-1)$ is an unitarizable representation $\widehat{L(\nu^{\frac{1}{2}}\delta_{n-1}; \sigma_{n-1})}$, in the corresponding Jantzen filtration (the same structure as for the index $n$) it has a signature $(0, q_1')$ and the quotient (the sum of the Aubert-duals of the square-integrable representations) has a signature $(p_0', q_0')$. But for $s < 1/2$ the representation $\nu^s\widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}}$ is unitarizable, and on the other hand, it has a signature given by $(p_0' + q_1', q_0')$ [22]. We conclude that $q_0' = 0$, so the operator $A(1/2, n-1)$ induces a definite form on the quotient $\nu^{1/2}\widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}}/\widehat{L(\nu^{1/2}\delta_{n-1}; \sigma_{n-1})}$.

We then have

$$\mathrm{meas}(K_{max,2})\mathrm{meas}(K_{max,1})L = \int_K (v_3^{(k_2)}, v_3^{(k_2)})A(k_2)C(k_2)\mathrm{d}k_2,$$

which is then, obviously, positive.

The (semi)-definiteness of the operator $A(1/2, n-1)$ also follows for $n-1 = 1$ if $\alpha > 1/2$ by the same proposition. So, we only have to analyze the case $n-1 = 1$ when $\alpha = 1/2$, the structure of the representation $\nu^{1/2}\widehat{\delta_{n-1}} \rtimes \widehat{\sigma_{n-1}}$ and the definiteness of the operator $A(1/2, n-1)$ in that case. We shall do that in the following lemma. ∎

Lemma 3.7: *Let $\alpha = 1/2$. Then, the representations $\widehat{L(\nu^{1/2}\rho; \sigma_1)}$ and $\widehat{\sigma_1'}$ are unitarizable, and the intertwining operator*

$$A(1/2, 1) : \nu^{1/2}\rho \rtimes \widehat{\sigma_1} \to \nu^{-1/2}\rho \rtimes \widehat{\sigma_1}$$

*is holomorphic, positive-semidefinite on the compact picture of the representation $\nu^{1/2}\rho \rtimes \widehat{\sigma_1}$. The representation $\widehat{\sigma_1}$ is unitarizable.*

*Proof.* We assume that $\rho$ is a representation of $GL(k, F)$ and $\sigma_{1,cusp} = \sigma'$ is a representation of $G_m$. By the previous considerations, we know that, in the Grothendieck group, we have

$$\nu^{1/2}\rho \rtimes \widehat{\sigma_1} = \widehat{L(\nu^{1/2}\rho; \sigma_1)} + \widehat{\sigma_1'}.$$

By the equalities in the discussion following the Proposition 3.1, the representation $\widehat{\sigma_1'}$ is unitarizable, because it appears at the end of the complementary series of certain unitarizable representation.

It is not difficult to see that $\widehat{\sigma_1'} = L(\rho\nu^{3/2}, \rho\nu^{1/2}, \rho\nu^{1/2}; \sigma')$. The representation $\delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \rtimes \sigma'$ reduces into a sum of two non-equivalent, tempered representations, say $\tau_1$ and $\tau_2$. When examining Jacquet modules, we see that for one of these representations, $r_{GL(2k,F)\times G_m}$-module is $\delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \otimes \sigma'$, and for the other is $\delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \otimes \sigma' + \nu^{1/2}\rho \times \nu^{1/2}\rho$. We call the first one $\tau_1$ and the second one $\tau_2$. When analyzing the representation $\nu^{1/2}\rho \times \nu^{1/2}\rho \rtimes \sigma'$, we see that then $\tau_1 \leq \nu^{1/2}\rho \rtimes L(\nu^{1/2}\rho; \sigma')$. Now, from, for example, [20, Theorem 8.1(ii)], it follows that $L(\widehat{\nu^{1/2}\rho; \sigma_1}) = L(\nu^{3/2}\rho; \tau_1)$. We analyze the representations $\nu^s\rho \rtimes \tau_1$, $s \geq 0$. We want to prove that $s = 3/2$ is the first positive reducibility point for this representation; it is not reducible for $s = 0$. Then, it would follow that the representation $L(\nu^{3/2}\rho; \tau_1)$ is unitarizable.

In order to do that, we shall prove that the representation $\nu^{1/2}\rho \rtimes \tau_1$ is irreducible. The representation $\tau_1$ is basic [14] and we can apply Lemma 6.1 and Lemma 6.2 of [14], and, in this situation, these lemmas can be stated as follows: Exactly one of the representations $\nu^{1/2}\rho \rtimes \tau_1$ and $\nu^{1/2}\rho \rtimes \tau_2$ reduces; call it $\tau$. Then, $\nu^{1/2}\rho \rtimes \tau = T_1 + L(\nu^{1/2}\rho; \tau)$, where $T_1 = \delta(\nu^{-1/2}\rho, \nu^{1/2}\rho) \rtimes \sigma_0$ (which is irreducible). We will show that $\tau = \tau_2$. Namely, if we assume that $T_1 \leq \nu^{1/2}\rho \rtimes \tau_1$, then also the corresponding relation must hold for the Jacquet modules. Using the formula (1) we obtain

$$\begin{aligned}\mu^*(T_1) &= \mu^*(\delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \rtimes \sigma_0) = M(\delta([\nu^{-1/2}\rho, \nu^{1/2}\rho)]) \rtimes \mu^*(\sigma_0)\\ &\leq M(\nu^{1/2}\rho) \rtimes \mu^*(\tau_1).\end{aligned}$$

When we compare $r_{GL(3k,F)\times G_m}$-Jacquet modules, we see that the subquotient $\nu^{1/2}\rho \times \nu^{1/2}\rho \times \nu^{1/2}\rho \otimes \sigma'$ appears on the left-hand side, and since

$$\begin{aligned}&r_{GL(3k,F)\times G_m}(\nu^{1/2}\rho \rtimes \tau_1)\\ &\quad = \nu^{-1/2}\rho \times \delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \otimes \sigma' + \nu^{1/2}\rho \times \delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \otimes \sigma',\end{aligned}$$

this cannot be true.

Now, again we calculate the Plancherel measure $\mu(s, \nu^s\rho \rtimes \widehat{\sigma_1})$. In this case, we can explicitly (without involving any conjectures) calculate the pole of the Plancherel measure: namely, we can use the result of Heirmann which states that when inducing from the cuspidal representation, and then considering the Plancherel measure attached to the long intertwining operator, if we have a discrete series subquotient, the pole of the Plancherel measure equals the corank of the parabolic subgroup involved. Again, we get that the Plancherel measure has a simple pole for $s = 1/2$. Since the representation $\nu^{1/2}\rho \rtimes \widehat{\sigma_1}$ has two unitarizable subquotients, a unique quotient, (namely, $\widehat{\sigma_1'}$) and a unique subrepresentation, we can use the Jantzen filtration in an analogous, but simpler way as in Proposition 3.6, to prove that the representation $\widehat{\sigma_1}$ is unitarizable. ∎

To conclude the proof, we shall have to address the holomorphy questions. Again, we prove the holomorphy of the intertwining operators in a bit more general context (than the Steinberg case).

Proposition 3.8: *Let $\sigma_1$ be a strongly positive discrete series representation of $G_m$, let $\rho$ be an irreducible, self-dual representation of $GL(k, F)$ and let $a_- > 2$ be a positive integer such that $a_- - 2,\ a_-,\ a_- + 4 \notin \mathrm{Jord}_\rho(\sigma_1)$ and $a_- + 2 \in \mathrm{Jord}_\rho(\sigma_1)$. Let $\widehat{\delta} = \delta(\widehat{[\nu^{-(a_-/2-1)}\rho, \nu^{a_-/2-1}\rho]})$. Let $\widehat{\sigma_2}$ be a unique subrepresentation of the representation $\nu^{-\frac{a_-+3}{2}}\rho \rtimes \widehat{\sigma_1}$ (Lemma 3.5). If we assume that the (standard) intertwining operator*

$$A(s, \widehat{\delta} \otimes \widehat{\sigma_1}) : \nu^s\widehat{\delta} \rtimes \widehat{\sigma_1} \to \nu^{-s}\widehat{\delta} \rtimes \widehat{\sigma_1}$$

*is holomorphic for $s = 1/2$, then the intertwining operator*

$$A(s, \widehat{\delta[\nu^{-a_-/2}\rho, \nu^{a_-/2}\rho]} \otimes \widehat{\sigma_2}, w) : \nu^s\widehat{\delta[\nu^{-a_-/2}\rho, \nu^{a_-/2}\rho]} \rtimes \widehat{\sigma_2} \to \nu^{-s}\widehat{\delta[\nu^{-a_-/2}\rho, \nu^{a_-/2}\rho]} \rtimes \widehat{\sigma_2}$$

*is holomorphic for $s = 1/2$. An element $w$ of the Weyl group is the one defined in the Proposition 3.6.*

*Proof.* Using the factorization of the intertwining operators [15] we get:

$$A\Big(s - \frac{a_-}{2}, s, s + \frac{a_-}{2}, -\frac{a_- + 3}{2}; \rho \otimes \widehat{\delta} \otimes \rho \otimes \rho, \widehat{\sigma_1}; w\Big) = B_1(s)A''(s)B_2(s),$$

for the intertwining operators $B_1(s)$, $A''(s)$ and $B_2(s)$ (essentially, the situation we had in the Proposition 3.6). Their actions are described as follows:

$$B_2(s) : \nu^{s-\frac{a_-}{2}}\rho \times \nu^s\widehat{\delta} \times \nu^{s+\frac{a_-}{2}}\rho \times \nu^{-\frac{a_-+3}{2}}\rho \rtimes \widehat{\sigma_1} \to \nu^{-s-\frac{a_-}{2}}\rho \times \nu^{s-\frac{a_-}{2}}\rho \times \nu^{-\frac{a_-+3}{2}}\rho \times \nu^s\widehat{\delta} \rtimes \widehat{\sigma_1},$$

$$A''(s) : \nu^{-s-\frac{a_-}{2}}\rho \times \nu^{s-\frac{a_-}{2}}\rho \times \nu^{-\frac{a_-+3}{2}}\rho \times \nu^{s}\widehat{\delta} \rtimes \widehat{\sigma_1} \to \nu^{-s-\frac{a_-}{2}}\rho \times \nu^{s-\frac{a_-}{2}}\rho \times \nu^{-\frac{a_-+3}{2}}\rho \times \nu^{-s}\widehat{\delta} \rtimes \widehat{\sigma_1}$$

and

$$B_1(s) : \nu^{-s-\frac{a_-}{2}}\rho \times \nu^{s-\frac{a_-}{2}}\rho \times \nu^{-\frac{a_-+3}{2}}\rho \times \nu^{-s}\widehat{\delta} \rtimes \widehat{\sigma_1} \to \nu^{-s-\frac{a_-}{2}}\rho \times \nu^{-s}\widehat{\delta} \times \nu^{-s+\frac{a_-}{2}}\rho \times \nu^{-\frac{a_-+3}{2}}\rho \rtimes \widehat{\sigma_1}.$$

When we factorize the operator $B_2(s)$ into the generalized rank one intertwining operators, it follows that we have to check the holomorphy of the intertwining operators

$$\widehat{C_2}(s) : \nu^{s}\widehat{\delta} \times \nu^{-s-\frac{a_-}{2}}\rho \to \nu^{-s-\frac{a_-}{2}}\rho \times \nu^{s}\widehat{\delta},$$

$$\widehat{D_2}(s) : \nu^{s}\widehat{\delta} \times \nu^{-\frac{a_-+3}{2}}\rho \to \nu^{-\frac{a_-+3}{2}}\rho \times \nu^{s}\widehat{\delta}$$

and

$$\widehat{E_2}(s) : \nu^{s+\frac{a_-}{2}}\rho \rtimes \widehat{\sigma_1} \to: \nu^{-(s+\frac{a_-}{2})}\rho \rtimes \widehat{\sigma_1}$$

near $s = 1/2$.

To prove the holomorphy of the operator $B_1(s)$ we have to check the holomorphy of the operators

$$\widehat{C_1}(s) : \nu^{-\frac{a_-+3}{2}}\rho \times \nu^{-s}\widehat{\delta} \to \nu^{-s}\widehat{\delta} \times \nu^{-\frac{a_-+3}{2}}\rho,$$

$$\widehat{D_1}(s) : \nu^{s-\frac{a_-}{2}}\rho \times \nu^{-s}\widehat{\delta} \to \nu^{-s}\widehat{\delta} \times \nu^{s-\frac{a_-}{2}}\rho$$

and

$$\widehat{E_1}(s) : \nu^{s-\frac{a_-}{2}}\rho \rtimes \widehat{\sigma_1} \to \nu^{-(s-\frac{a_-}{2})}\rho \rtimes \widehat{\sigma_1}$$

near $s = 1/2$.

We consider the intertwining operator $\widehat{C_2}(s)$ near $s = 1/2$. We can embed $\nu^{1/2}\widehat{\delta} \hookrightarrow \nu^{-(a_-/2-3/2)}\rho \times \cdots \times \nu^{a_-/2-1/2}\rho$ and when we factorize an appropriate intertwining operator on this larger space, we get holomorphy. The same procedure works for $\widehat{D_2}(1/2)$ and $\widehat{C_1}(1/2)$. For $\widehat{D_1}(1/2)$ we have to proceed in a slightly different manner, since $\nu^{1/2-a_-/2}\rho$ appears in the cuspidal support of

$\delta([\widehat{\nu^{-(a_-/2-1/2)}\rho, \nu^{a_-/2-3/2}}\rho])$. We can embed

$$\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho \times \delta([\widehat{\nu^{-(\frac{a_-}{2}-\frac{1}{2})}\rho, \nu^{\frac{a_-}{2}-\frac{3}{2}}}\rho]) \hookrightarrow$$
$$\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho \times \delta([\widehat{\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho, \nu^{\frac{3}{2}-\frac{a_-}{2}}}\rho]) \times \delta([\widehat{\nu^{\frac{5}{2}-\frac{a_-}{2}}\rho, \nu^{\frac{a_-}{2}-\frac{3}{2}}}\rho]).$$

So, we have to prove the holomorphy of the operator

$$\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho \times \delta([\widehat{\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho, \nu^{\frac{3}{2}-\frac{a_-}{2}}}\rho]) \to \delta([\widehat{\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho, \nu^{\frac{3}{2}-\frac{a_-}{2}}}\rho]) \times \nu^{\frac{1}{2}-\frac{a_-}{2}}\rho$$

The operator

$$\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho \times \delta([\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho, \nu^{\frac{3}{2}-\frac{a_-}{2}}\rho]) \to \delta([\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho, \nu^{\frac{3}{2}-\frac{a_-}{2}}\rho]) \times \nu^{\frac{1}{2}-\frac{a_-}{2}}\rho$$

has a simple pole (for $s = 1/2$). Indeed, this follows from the calculation of the Plancherel measure and the fact that the operator

$$\delta([\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho, \nu^{\frac{3}{2}-\frac{a_-}{2}}\rho]) \times \nu^{\frac{1}{2}-\frac{a_-}{2}}\rho \to \nu^{\frac{1}{2}-\frac{a_-}{2}}\rho \times \delta([\nu^{\frac{1}{2}-\frac{a_-}{2}}\rho, \nu^{\frac{3}{2}-\frac{a_-}{2}}\rho])$$

is a holomorphic isomorphism on the irreducible representation space. Since on the Aubert-dual side we have the same Plancherel measure and we have a pole of the appropriate intertwining operator acting (essentially) on the space $\nu^{-1/2}\rho \times \nu^{1/2}\rho \times \nu^{-1/2}\rho$, we get the holomorphy of the operator in question.

The representations $\nu^{a_-/2\pm 1/2}\rho \rtimes \sigma_1$ are irreducible [12]. When we calculate the corresponding Plancherel measures (like in Lemma 3.2), we obtain the holomorphy of the operator $\widehat{E_1(1/2)}$. The operator $\widehat{E_2(s)}$ has a simple pole for $s = 1/2$. On the other hand, the following holds:

$$\nu^{-\frac{a_-+1}{2}}\rho \rtimes \widehat{\sigma_2} \hookrightarrow L(\nu^{-\frac{a_-+1}{2}}\rho, \nu^{-\frac{a_-+3}{2}}\rho) \rtimes \widehat{\sigma_1}.$$

We will denote by $T(s)$ an operator which occurs in the factorization of the operator $B_2(s)$ and which acts after the operator induced from $\widehat{E_2}(s)$, i.e. $B_2(s) = T(s)\widehat{E_2}(s)$. From the previous relation follows that $T(s)$ has the following property: the representation $\delta[\widehat{\nu^{-a_-/2}\rho, \nu^{a_-/2}}\rho] \rtimes \widehat{\sigma_2}$ is in its kernel (for $s = 1/2$). So, for $f$ in the compact picture of the representation $\delta[\widehat{\nu^{-a_-/2}\rho, \nu^{a_-/2}}\rho] \rtimes \widehat{\sigma_2}$, we observe that the function $T(s)\widehat{E_2}(s)f_s$ is holomorphic near $s = 1/2$, so $B_2(s)$, when restricted to the representation space of the representation $\delta[\widehat{\nu^{-a_-/2}\rho, \nu^{a_-/2}}\rho] \rtimes \widehat{\sigma_2}$, is holomorphic. ∎

By Proposition 3.8 we have reduced the question of the holomorphy of the intertwining operators appearing in the generalized Steinberg case to the following situation:

Lemma 3.9: *Let $\rho$ be an irreducible, selfcontragredient, supercuspidal representation of the group $GL(k,F)$ and let $\sigma'$ be a similar representation of $G_m$. Assume that the representation $\nu^\alpha\rho\rtimes\sigma'$ reduces and $\alpha>0$. Then, the operator*

$$A(\frac{1}{2},\widehat{\delta_1}\otimes\widehat{\sigma_1}):\delta([\nu^{-\widehat{(\alpha-1)}\rho,\nu^\alpha}\rho])\rtimes\widehat{\sigma_1}\rightarrow\delta([\nu^{-\alpha}\widehat{\rho,\nu^{(\alpha-1)}}\rho])\rtimes\widehat{\sigma_1}$$

*is holomorphic.*

*Proof.* We embed

$$\delta([\nu^{-\widehat{(\alpha-1)}\rho,\nu^\alpha}\rho])\rtimes\widehat{\sigma_1}\hookrightarrow\delta([\nu^{-\widehat{(\alpha-1)}\rho,\nu^\alpha}\rho])\times\nu^{-(\alpha+1)}\rho\rtimes\widehat{\sigma_0}.$$

We have already covered the case $\alpha=1/2$, so, we assume $\alpha\geq 1$. When we factorize the appropriate intertwining operator acting on the representation above (on the right-hand side), we see that the poles of the generalized rank one case intertwining operators cancel with the kernels of the other operators appearing (similarly as in the Proposition 3.8). Namely,

$$\delta([\nu^{-\widehat{(\alpha-1)}\rho,\nu^\alpha}\rho])\rtimes\widehat{\sigma_1}=L(\delta([\nu^{-\widehat{(\alpha-1)}\rho,\nu^\alpha}\rho]);\sigma_1)+\widehat{\sigma_1'}+\widehat{\sigma_2'},$$

where $\sigma_1'$ and $\sigma_2'$ are square -integrable representations. On the other hand, the kernel of the (holomorphic) intertwining operator

$$\nu^{-(\alpha+1)}\rho\times\delta([\nu^{-\widehat{\alpha}\rho,\nu^{\alpha}}{}^{-1}\rho])\rtimes\widehat{\sigma_0}\rightarrow\delta([\nu^{-\widehat{\alpha}\rho,\nu^{\alpha}}{}^{-1}\rho])\times\nu^{-(\alpha+1)}\rho\rtimes\widehat{\sigma_0}$$

is the representation $\delta([\nu^{-\widehat{(\alpha+1)}\rho,\nu^{\alpha-1}}\rho])\rtimes\widehat{\sigma_0}$ which reduces according to [12, Theorem 4.1]. On the other hand, since we know that the representation $\widehat{\sigma_0}$ is unitarizable, there is a positive definite hermitian form on the space $\nu^s\delta([\nu^{-\widehat{(\alpha-1/2)}\rho,\nu^{\alpha-1/2}}\rho])\rtimes\widehat{\sigma_0}$ for $s\in[0,1/2)$ and by examining the Jantzen filtration we see that the pole of the intertwining operator is of order one. ∎

Corollary 3.10: *Let $\rho$ be an irreducible, selfcontragredient, supercuspidal representation of the group $GL(k,F)$ and let $\sigma'$ be a similar representation of $G_m$. Assume that the representation $\nu^\alpha\rho\rtimes\sigma'$ reduces and $\alpha>0$. Then, for each $n\geq 1$, the Aubert dual of the generalized Steinberg representation $\sigma_n$, i.e., the representation $L(\nu^{n+\alpha}\rho,\nu^{n-1+\alpha}\rho,\ldots,\nu^\alpha\rho;\sigma')$, is unitarizable.*

*Proof.* By our inductive procedure, the proof follows as soon as the basis for the induction is proved. For $\alpha>1/2$ it is enough to consider the representations $\widehat{\sigma_{-1}}=\widehat{\sigma'}=\sigma'$ and $\widehat{\sigma_0}=L(\nu^{1/2}\rho;\sigma')$ which are trivially unitarizable; for $\alpha=1/2$ we proved the unitarizability of $\widehat{\sigma_1}$ in Lemma 3.7. ∎

## 4. Strongly positive discrete series in $\mathcal{D}(\rho, \sigma')$

In this section we deal with the case of the strongly positive discrete series belonging to the set $\mathcal{D}(\rho, \sigma')$ [7], i.e., those whose partial supercuspidal support is a supercuspidal representation $\sigma'$, and the rest of the supercuspidal support is formed from the twists of the representation $\rho$. Of course, we continue to assume $\rho \cong \tilde{\rho}$.

If the representation $\rho \rtimes \sigma'$ reduces, then the only strongly positive discrete series in $\mathcal{D}(\rho, \sigma')$ is $\sigma'$.

Assume that for some $\alpha > 0$ (we assume that $\alpha \in \frac{1}{2}\mathbb{Z}$) the representation $\rho\nu^{\alpha} \rtimes \sigma'$ reduces. Then

$$\mathrm{Jord}_{\rho}(\sigma') = \begin{cases} \{1, 3, 5, \dots, 2\alpha - 1\}, & \text{if } \alpha \in \mathbb{Z}, \\ \{2, 4, \dots, 2\alpha - 1\}, & \text{if } \alpha \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z},\ \alpha \neq \frac{1}{2}, \\ \emptyset, & \text{if } \alpha = 1/2. \end{cases}$$

Then, for strongly positive $\sigma \in \mathcal{D}(\rho, \sigma')$ and $\sigma \neq \sigma'$, the Jordan block is

$$\mathrm{Jord}_{\rho}(\sigma) = \begin{cases} \{a_1, a_2, \dots, a_{\alpha}\}, & \text{if } \alpha \in \mathbb{Z}, \\ \{a_1, a_2, \dots, a_{\alpha - \frac{1}{2}}\}, & \text{if } \alpha \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z},\ \epsilon(a_1) = -1, \\ \{a_1, a_2, \dots, a_{\alpha + \frac{1}{2}}\}, & \text{if } \alpha \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z},\ \epsilon(a_1) = 1. \end{cases}$$

We will now prove the unitarizability of the representations $\widehat{\sigma}$, for which $\alpha \in \mathbb{Z}$ or $\epsilon(a_1) = -1$.

We will do that using the two-fold inductive procedure, quite analogous to the one introduced in the generalized Steinberg representations case.

The first induction will be over the place of an element in the Jordan block; the second will be over the value of that element. More precisely, the basis for the first induction is the case of the generalized Steinberg representation covered in the previous section; in this case $\mathrm{Jord} = \{2, 4, \dots, 2\alpha - 3, a_{\alpha - 1/2}\}$ (or $\mathrm{Jord} = \{1, 3, \dots, 2\alpha - 3, a_{\alpha}\}$; we have actually completely covered the case of $\alpha = 1/2$ by the generalized Steinberg representations, so we assume $\alpha \geq 1$).

So, the first induction starts from the largest element in the Jordan block (so for this element we put $i = 1$). By the inductive hypothesis (of the first induction) we assume that the Aubert dual of the representation with Jordan block

$$\mathrm{Jord}(\sigma_i) = \{2, 4, \dots, 2j, a_{j+1}, \dots, a_{\alpha - \frac{1}{2}}\}$$

or

$$\mathrm{Jord}(\sigma_i) = \{1, 3, \ldots, 2j-1, a_{j+1}, \ldots, a_\alpha\}$$

is unitarizable. Here, of course, $\alpha - 1/2 - i = j$ (or $\alpha - j = i$). We want to prove that the Aubert dual of the representation whose Jordan block equals to

$$\mathrm{Jord}(\sigma_{i+1}) = \{2, 4, \ldots, 2j-2, a_j, a_{j+1}, \ldots, a_{\alpha-\frac{1}{2}}\}$$

or

$$\mathrm{Jord}(\sigma_{i+1}) = \{1, 3, \ldots, 2j-3, a_j, a_{j+1}, \ldots, a_\alpha\}$$

is unitarizable. We continue this induction over the $i$'s until we get such $j$ for which $a_j = 2j$ (or $a_j = 2j-1$). So, we put $a_j = 2j+2n_{i+1}$ (or $a_j = 2j-1+2n_{i+1}$) only now we use the induction over $n_{i+1}$ (the second induction), and accordingly, introduce representations $\sigma_{i+1,n_{i+1}}$ (for which $a_j = 2j + 2n_{i+1}$ in the Jordan block). For $n_{i+1} = 0$ we have $a_j = 2j$ (or $a_j = 2j - 1$) and, we obtain the representation $\sigma_i$ from the inductive hypothesis (of the first induction). Of course, $n_{i+1}$ is such that $2j + 2n_{i+1} < a_{j+1}$ (or $2j - 1 + 2n_{i+1} < a_{j+1}$).

We examine the situation for $n_{i+1} = 1$. Using the results from [12], we immediately see that the first point of reducibility of the representation $\nu^s\rho \rtimes \sigma_i$ is $s = j + 1/2$ (or $s = j$), and $\sigma_{i+1,1} \hookrightarrow \nu^{j+1/2}\rho \rtimes \sigma_i$ (or $\sigma_{i+1,1} \hookrightarrow \nu^j\rho \rtimes \sigma_i$). Actually, we can take $s = \alpha - i$ in both cases. From this immediately follows that the representation $\widehat{\sigma_{i+1,1}}$ is unitarizable.

Now we proceed in a fashion which is quite analogous to the generalized Steinberg case from the previous section.

For $n_{i+1} \geq 2$, we examine the following representations:

$$\nu^{\frac{1}{2}}\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-\frac{3}{2})}\rho, \nu^{(n_{i+1}+\alpha-(i+1)-\frac{3}{2})}\rho]) \rtimes \sigma_{i+1,n_{i+1}},$$
$$\nu\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-1)}\rho, \nu^{(n_{i+1}+\alpha-(i+1)-1)}\rho]) \rtimes \sigma_{i+1,n_{i+1}-1},$$
$$\nu^{\frac{1}{2}}\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-\frac{1}{2})}\rho, \nu^{(n_{i+1}+\alpha-(i+1)-\frac{1}{2})}\rho]) \rtimes \sigma_{i+1,n_{i+1}-2}.$$

We get

$$\nu^{\frac{1}{2}}\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-\frac{3}{2})}\rho, \nu^{(n_{i+1}+\alpha-(i+1)-\frac{3}{2})}\rho]) \rtimes \sigma_{i+1,n_{i+1}} = L(\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-2)}\rho, \nu^{(n_{i+1}+\alpha-(i+1)-1)}\rho]); \sigma_{i+1,n_{i+1}}) + \sigma_1' + \sigma_2',$$

$$\begin{aligned}\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-2)}\rho,\nu^{n_{i+1}+\alpha-(i+1)}\rho])\rtimes\sigma_{i+1,n_{i+1}-1} = \\ \sigma_1''+\sigma_2''+L(\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-2)}\rho,\nu^{n_{i+1}+\alpha-(i+1)}\rho]);\sigma_{i+1,n_{i+1}-1}) \\ +L(\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-2)}\rho,\nu^{(n_{i+1}+\alpha-(i+1)-1)}\rho]);\sigma_{i+1,n_{i+1}}) \\ +L(\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-1)}\rho,\nu^{(n_{i+1}+\alpha-(i+1))}\rho]);\sigma_{i+1,n_{i+1}-2}),\end{aligned}$$

$$\begin{aligned}\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-1)}\rho,\nu^{n_{i+1}+\alpha-(i+1)}\rho])\rtimes\sigma_{i+1,n_{i+1}-2} = \\ L(\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-1)}\rho,\nu^{n_{i+1}+\alpha-(i+1)}\rho]);\sigma_{i+1,n_{i+1}-2})+\sigma_1'''+\sigma_2'''.\end{aligned}$$

The square integrable representations $\sigma_1'$ and $\sigma_2'$ appear also as the subquotients in the second and the third induced representation above, so we may take $\sigma_1'=\sigma_1''$ and $\sigma_2'=\sigma_2'''$; then $\sigma_2''=\sigma_1'''$. We will denote a discrete series representation of the general linear group appearing in the first induced representation by $\delta_{i+1,n_{i+1}}$.

We can now state the following

Proposition 4.1: *Assume that, for $n_{i+1}\geq 2$, the representations $\sigma_{\widehat{i+1,n_{i+1}-1}}$ and $\sigma_{\widehat{i+1,n_{i+1}-2}}$ are unitarizable. Then the representation $\sigma_{\widehat{i+1,n_{i+1}}}$ is unitarizable.*

*Proof.* By the considerations in this section, assuming the unitarizability of the representations $\sigma_{\widehat{i+1,n_{i+1}-1}}$ and $\sigma_{\widehat{i+1,n_{i+1}-2}}$, all the irreducible subquotients of the representation $\nu^{1/2}\delta_{\widehat{i+1,n_{i+1}}}\rtimes\sigma_{i+1,n_{i+1}}$ are unitarizable. From the fact that

$$\sigma_{i+1,n_{i+1}}\hookrightarrow\rho\nu^{n_{i+1}+\alpha-(i+1)}\rtimes\sigma_{i+1,n_{i+1}-1}$$

and from Lemma 3.5, we see that the following holds

$$\sigma_{\widehat{i+1,n_{i+1}}}\hookrightarrow\rho\nu^{-(n_{i+1}+\alpha-(i+1))}\rtimes\sigma_{\widehat{i+1,n_{i+1}-1}}.$$

From this it follows that

$$\begin{aligned}\nu^s\delta_{\widehat{i+1,n_{i+1}}}\rtimes\sigma_{i+1,n_{i+1}}\hookrightarrow\nu^{s-(n_{i+1}+\alpha-(i+1)-3/2)}\rho\times\nu^s\delta_{\widehat{i+1,n_{i+1}-1}} \\ \times\nu^{s+(n_{i+1}+\alpha-(i+1)-3/2)}\rho\times\nu^{-(n_{i+1}+\alpha-(i+1))}\rho\rtimes\sigma_{\widehat{i+1,n_{i+1}-1}}.\end{aligned}\tag{18}$$

Now, we again examine the corresponding intertwining operator

$$A(s,i+1,n_{i+1}):\nu^s\delta_{\widehat{i+1,n_{i+1}}}\rtimes\sigma_{i+1,n_{i+1}}\rightarrow\nu^{-s}\delta_{\widehat{i+1,n_{i+1}}}\rtimes\sigma_{i+1,n_{i+1}}\tag{19}$$

by embedding the representation on the left-hand side of the previous expression as in (18), and analyzing the form induced by the appropriate intertwining operator on $\nu^{s-(n_{i+1}+\alpha-(i+1)-3/2)}\rho \times \nu^s \widehat{\delta_{i+1,n_{i+1}-1}} \times \nu^{s+(n_{i+1}+\alpha-(i+1)-3/2)}\rho \times \nu^{-(n_{i+1}+\alpha-(i+1))}\rho \rtimes \sigma_{\widehat{i+1,n_{i+1}}-1}$.

We can now completely follow the proof of Proposition 3.6 while proving that the form on $\nu^{1/2}\widehat{\delta_{i+1,n_{i+1}}} \rtimes \sigma_{i+1,n_{i+1}}$ induced by the operator $A(\frac{1}{2}, i+1, n_{i+1})$, is of the right signature. We use Lemmas 3.2, 3.3 and Proposition 3.8. ∎

We now settle the case $\alpha \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \alpha \neq 1/2$ and $\epsilon(a_1) = 1$.

We start with the following situation:

$$\mathrm{Jord}(\sigma_1) = \{2, 4, \ldots, 2\alpha - 1, a_{\alpha+1/2}\}.$$

We want to prove that $\widehat{\sigma_1}$ is unitarizable. So, again, we put $a_{\alpha+1/2} = 2\alpha + 1 + 2n_1$. For $n_1 = 0$ we get the representation $\sigma_{1,0}$ which is a unique subrepresentation of the induced representation

$$\nu^{1/2}\rho \times \nu^{3/2}\rho \times \cdots \nu^{\alpha-1}\rho \times \nu^{\alpha}\rho \rtimes \sigma'.$$

Lemma 4.2: *The representation $\widehat{\sigma_{1,0}}$ is unitarizable.*

*Proof.* The representation $\widehat{\sigma_{1,0}}$ is a unique quotient of the representation $\nu^{1/2}\rho \times \nu^{3/2}\rho \times \cdots \nu^{\alpha-1}\rho \times \nu^{\alpha}\rho \rtimes \sigma'$. The representation $\widehat{\sigma_{1,0}}$ is a unique subquotient of $\nu^{1/2}\rho \times \nu^{3/2}\rho \times \cdots \nu^{\alpha-1}\rho \times \nu^{\alpha}\rho \rtimes \sigma'$ whose $r_{GL(k,F)^{\alpha+1/2} \times G_m}$–Jacquet module contains $\nu^{-1/2}\rho \otimes \nu^{-3/2}\rho \otimes \cdots \nu^{-(\alpha-1)}\rho \otimes \nu^{-\alpha}\rho \otimes \sigma'$ as a subquotient ([3], Lemma 4.1). We see that the representation $L(\delta([\nu^{1/2}\rho, \nu^{\alpha}\rho]); \sigma')$ satisfies the required condition, so $\widehat{\sigma_{1,0}} = L(\delta([\nu^{1/2}\rho, \nu^{\alpha}\rho]); \sigma')$. We examine the representation $\nu^s \delta([\nu^{-\frac{\alpha-1/2}{2}}\rho, \ldots, \nu^{\frac{\alpha-1/2}{2}}\rho]) \rtimes \sigma',\ s \geq 0$. Examining the cases $\frac{\alpha-1/2}{2} \in \mathbb{Z}$ and $\frac{\alpha-1/2}{2} \notin \mathbb{Z}$ separately, we conclude that, in both of these cases, the first reducibility point is $s = \frac{\alpha+1/2}{2}$. But $\nu^{\frac{\alpha+1/2}{2}}\delta([\nu^{-\frac{\alpha-1/2}{2}}\rho, \nu^{\frac{\alpha-1/2}{2}}\rho]) \rtimes \sigma' = \delta([\nu^{1/2}\rho, \nu^{\alpha}\rho]) \rtimes \sigma'$, so the representation $L(\delta([\nu^{1/2}\rho, \nu^{\alpha}\rho]); \sigma')$ is unitarizable. ∎

Lemma 4.3: *The representation $\widehat{\sigma_{1,1}}$ is unitarizable.*

*Proof.* Let $\sigma_1'$ be a strongly positive discrete series representation which belongs to $D(\rho, \sigma')$ and such that $\mathrm{Jord}(\sigma_1') = \{4, 6, \ldots 2\alpha - 1, 2\alpha + 3\}$. We also note that, necessarily, $\epsilon_{\sigma_1'}(4) = -1$. We note that $\nu^{1/2}\rho \rtimes \sigma_1' = L(\nu^{1/2}\rho; \sigma_1) + \sigma_{1,1}$. Since we have already proved that the representations such as $\widehat{\sigma_1'}$ are unitarizable, it follows that $\widehat{\sigma_{1,1}}$ is unitarizable. ∎

We use the same procedure to prove the unitarizability of the representations $\sigma_{1,n_1}$ in this setting, as we did in the generalized Steinberg case; we just have to alter a bit the discrete series representations of general linear groups appearing in the induced representations we introduce.

Proposition 4.4: *Assume $n_1 \geq 2$. Then, if we assume the unitarizability of the representations $\widehat{\sigma_{1,n_1-1}}$ and $\widehat{\sigma_{1,n_1-2}}$, all the irreducible subquotients of the representation*

$$\nu^{\frac{1}{2}}\delta([\nu^{-(n_1+\alpha-3/2)}\rho, \dots \nu^{n_1+\alpha-3/2}\rho]) \rtimes \widehat{\sigma_{1,n_1}}$$

*are unitarizable.*

*Proof.* We compare the composition series of the representations

$$\nu^{\frac{1}{2}}\delta([\nu^{-(n_1+\alpha-\frac{3}{2})}\rho, \dots \nu^{n_1+\alpha-\frac{3}{2}}\rho]) \rtimes \widehat{\sigma_{1,n_1}},$$
$$\nu\delta([\nu^{-(n_1+\alpha-1)}\rho, \dots \nu^{n_1+\alpha-1}\rho]) \rtimes \widehat{\sigma_{1,n_1-1}},$$
$$\nu^{\frac{1}{2}}\delta([\nu^{-(n_1+\alpha-\frac{1}{2})}\rho, \dots \nu^{n_1+\alpha-\frac{1}{2}}\rho]) \rtimes \widehat{\sigma_{1,n_1-2}}. \quad \blacksquare$$

We now treat the general case of the strongly positive discrete series of this kind. We proceed with the same kind of the two-fold induction procedure. We want to show that the Aubert dual of the discrete series with the Jordan block

$$\mathrm{Jord} = \{a_1, a_2, \dots, a_{\alpha+1/2}\}$$

is unitarizable. Assume that this is true for a square-integrable representation $\sigma_i$ with the Jordan block

$$\mathrm{Jord}(\sigma_i) = \{2, 4, \dots, 2j, a_{j+1}, \dots a_{\alpha+1/2}\}.$$

In the expression above $i$ satisfies $\alpha + 1/2 - i = j$. For $i = 1$ we have already proved the statement. Let $\sigma_{i+1,n_{i+1}}$ be a representation with the Jordan block

$$\mathrm{Jord}(\sigma_{i+1,n_{i+1}}) = \{2, 4, \dots, 2j + 2n_{i+1}, a_{j+1}, \dots a_{\alpha+1/2}\}.$$

For $n_{i+1} = 0$ we get representation $\sigma_i$, so, in this case, $\widehat{\sigma_{i+1,n_{i+1}}}$ is unitarizable by the inductive hypothesis. For $n_{i+1} = 1$ we get unitarizability similarly as in the previous Lemma.

Lemma 4.5: *The representation $\widehat{\sigma_{i+1,1}}$ is unitarizable.*

*Proof.* Let $\sigma_1'$ be a strongly positive discrete series in $\mathcal{D}(\rho, \sigma')$ such that

$$\mathrm{Jord}(\sigma_1') = \{4, \dots, 2j + 2, a_{j+1}, \dots, a_{\alpha+1/2}\};$$

obviously $\epsilon(\min(\mathrm{Jord}(\sigma_1')) = -1$. Observe that

$$\nu^{1/2}\rho \rtimes \sigma_1' = L(\nu^{1/2}\rho; \sigma_1') + \sigma_{i+1,1}.$$

Since $s = 1/2$ is the first point of reducibility of the representation $\nu^s\rho \rtimes \widehat{\sigma_1'}$, the claim follows. ∎

We continue in the same manner as with the strongly positive discrete series representations with $|\mathrm{Jord}| = \alpha - 1/2$; we just have to adjust essentially square-integrable representations by which we form induced representations.

For $n_{i+1} \geq 2$, we analyze the following representations:

$$\nu^{\frac{1}{2}}\delta([\nu^{-(n_{i+1}+\alpha-(i+1)-\frac{1}{2})}\rho, \nu^{(n_{i+1}+\alpha-(i+1)-\frac{1}{2})}\rho]) \rtimes \sigma_{i+1,n_{i+1}},$$
$$\nu\delta([\nu^{-(n_{i+1}+\alpha-(i+1))}\rho, \nu^{(n_{i+1}+\alpha-(i+1))}\rho]) \rtimes \sigma_{i+1,n_{i+1}-1},$$
$$\nu^{\frac{1}{2}}\delta([\nu^{-(n_{i+1}+\alpha-(i+1)+\frac{1}{2})}\rho, \nu^{(n_{i+1}+\alpha-(i+1)+\frac{1}{2})}\rho]) \rtimes \sigma_{i+1,n_{i+1}-2}.$$

Proposition 4.6: *Assume that, for $n_{i+1} \geq 2$, the representations $\sigma_{\widehat{i+1,n_{i+1}}-1}$ and $\sigma_{\widehat{i+1,n_{i+1}}-2}$ are unitarizable. Then the representation $\sigma_{\widehat{i+1,n_{i+1}}}$ is unitarizable.*

*Proof.* We employ the same procedures as before. ∎

## 5. Strongly positive discrete series in general

Let $\sigma$ be a strongly positive discrete series representation of the group $G_m$. Let $\rho_1, \rho_2, \ldots, \rho_n$ be the self-contragredient, irreducible, non-isomorphic cuspidal representations of the general linear groups, and let $\sigma'$ be a cuspidal representation of a $G_{m'}$ such that $\sigma \in \mathcal{D}(\rho_1, \rho_2, \ldots, \rho_n; \sigma')$ (we take $n$ to be minimal). From this we conclude that for an irreducible, self–contragredient representation $\rho$ of a general linear group the following holds: if $\rho \notin \{\rho_1, \rho_2, \ldots, \rho_n\}$, then $\mathrm{Jord}_\rho(\sigma) = \mathrm{Jord}_\rho(\sigma')$. Now, we define a sequence $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the strongly positive discrete series representations as follows: $\sigma_k \in \mathcal{D}(\rho_1, \ldots \rho_k; \sigma')$; if $\rho \notin \{\rho_1, \rho_2, \ldots \rho_k\}$, $\mathrm{Jord}_\rho(\sigma_k) = \mathrm{Jord}_\rho(\sigma')$, and if $\rho \in \{\rho_1, \rho_2, \ldots \rho_k\}$, then $\mathrm{Jord}_\rho(\sigma_k) = \mathrm{Jord}_\rho(\sigma)$.

More precisely, if $\mathrm{Jord}_{\rho_l}(\sigma) = \{a_1^{(\rho_l)}, \ldots, a_{j_{\rho_l}}^{(\rho_l)}\}$, then the representation $\sigma$ is a unique subrepresentation of the representation

$$\times_{l=1}^{n} \times_{k=1}^{j_{\rho_l}} \delta([\nu^{\frac{\phi_l(a_k^{(\rho_l)})+1}{2}}\rho_l, \nu^{\frac{a_k^{(\rho_l)}-1}{2}}\rho_l]) \rtimes \sigma',$$

and $\sigma_k$ is a unique subrepresentation of

$$\times_{l=1}^{k} \times_{s=1}^{j_{\rho_l}} \delta([\nu^{\frac{\phi_l(a_s^{(\rho_l)})+1}{2}}\rho_l, \nu^{\frac{a_s^{(\rho_l)}-1}{2}}\rho_l]) \rtimes \sigma'.$$

We will prove the unitarizability of $\widehat{\sigma}$ using induction over $k = 1, 2, \ldots n$.

As for $k = 1$, in the previous section we have proved that the representation $\widehat{\sigma_1}$ is unitarizable ($\sigma_1 \in \mathcal{D}(\rho_1; \sigma')$).

We assume now that the representation $\widehat{\sigma_k}$ is unitarizable; we will prove that the representation $\widehat{\sigma_{k+1}}$ is unitarizable.

Now, we employ the same strategy as in the previous section; we will just briefly go through the main points.

Let $\mathrm{Jord}_{\rho_{k+1}}(\sigma_{k+1}) = \{a_1, a_2, \ldots, a_{t_{k+1}}\}$. Here, $t_{k+1}$ is obtained as follows: $\alpha_{k+1}$ is a positive number such that $\nu^{\alpha_{k+1}}\rho_{k+1} \rtimes \sigma'$ reduces; so $t_{k+1} = \alpha_{k+1}$ or $\alpha_{k+1} - 1/2$ or $\alpha_{k+1} + 1/2$ (obviously, if $\alpha_{k+1} = 1/2$ then $t_{k+1} = 1$). We introduce auxiliary strongly positive discrete series which will be of use for us for the double inductive procedure. We define $\sigma_{k+1,i}$ as follows: for $\rho \neq \rho_{k+1}$ $\mathrm{Jord}_\rho(\sigma_{k+1,i}) = \mathrm{Jord}_\rho(\sigma_k)$ and $\mathrm{Jord}_{\rho_{k+1}}(\sigma_{k+1,i}) = \{2, 4, \ldots, 2j, a_{j+1}, \ldots, a_{t_{k+1}}\}$, where $t_{k+1} - i = j$.

Also we define $\sigma_{k+1,i+1,n_{i+1}}$ as follows: for $\rho \neq \rho_{k+1}$, $\mathrm{Jord}_\rho(\sigma_{k+1,i+1,n_{i+1}}) = \mathrm{Jord}_\rho(\sigma_k)$ and $\mathrm{Jord}_{\rho_{k+1}}(\sigma_{k+1,i+1,n_{i+1}}) = \{2, 4, \ldots, 2j + 2n_{i+1}, a_{j+1}, \ldots, a_{t_{k+1}}\}$.

The first induction is over $i$, i.e. over the representations $\sigma_{k+1,i}$. For $i = 1$ we have the representation $\sigma_{k+1,1}$ for which $\mathrm{Jord}_{\rho_{k+1}}(\sigma_{k+1,1}) = \{2, 4, \ldots, 2j, \ldots, 2\alpha_{k+1} - 3, a_{t_{k+1}}\}$ (or $\mathrm{Jord}_{\rho_{k+1}}(\sigma_{k+1,1}) = \{1, \ldots, 2\alpha_{k+1} - 3, a_{t_{k+1}}\}$ or $\mathrm{Jord}_{\rho_{k+1}}(\sigma_{k+1,1}) = \{2, \ldots, 2\alpha_{k+1} - 1, a_{t_{k+1}}\}$). The first two cases are quite analogous to the generalized Steinberg case (Section 3); namely, since the reducibility points and the composition series of the induced representations of the form $\delta([\nu^{-l_1}\rho_{n+1}, \nu^{l_2}\rho_{n+1}]) \rtimes \sigma_k$ are completely governed by $\mathrm{Jord}_{\rho_{k+1}}(\sigma_k)$, and $\mathrm{Jord}_{\rho_{k+1}}(\sigma_k) = \mathrm{Jord}_{\rho_{k+1}}(\sigma')$, we get totally analogous composition series of the representation; the calculation of the signatures is also very similar. We just have to check the bases for the various inductive procedures we use. The same holds for the case $\mathrm{Jord}_{\rho_{k+1}}(\sigma_{k+1,1}) = \{2, \ldots, 2\alpha_{k+1} - 1, a_{t_{k+1}}\}$. First, we prove the generalization of Lemma 3.7.

Lemma 5.1: *Using the notation introduced in this section, assume*

$$\alpha_{k+1} = 1/2, \quad \mathrm{Jord}_{\rho_{k+1}}(\sigma') = \emptyset.$$

*Then, the representation $\widehat{\sigma_{k+1,1,2}}$ is unitarizable.*

Before we prove this lemma, we need the following result

Lemma 5.2: *The representation $\widehat{\sigma_{k+1,1,2}}$ is a unique irreducible quotient of the representation $L(\nu^{3/2}\rho_{k+1}, \nu^{1/2}\rho_{k+1}) \rtimes \widehat{\sigma_k}$.*

*Proof.* We denote $\rho = \rho_{k+1}$. By the Frobenius reciprocity and the expression for the appropriate Jacquet module of the Aubert dual of a representation (9), we see that it is sufficient to prove that the multiplicity of the irreducible subquotient $\delta([\nu^{1/2}\rho, \nu^{3/2}\rho]) \otimes \sigma_k$ in $\mu^*(\delta([\nu^{1/2}\rho, \nu^{3/2}\rho]) \rtimes \sigma_k)$ is equal to one. We have the following formula

$$\begin{aligned}\mu^*(\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_k) =&(\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \otimes 1 + \delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]) \otimes 1 \\ &+ \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{3}{2}} \otimes 1 + \nu^{\frac{3}{2}}\rho \otimes \nu^{\frac{1}{2}}\rho + \nu^{-\frac{1}{2}}\rho \otimes \nu^{\frac{3}{2}}\rho \\ &+ 1 \otimes \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho])) \rtimes \mu^*(\sigma_k).\end{aligned}$$

If we assume that $\pi_1 \otimes \pi_2$ is an irreducible subquotient of $\mu^*(\sigma_k)$ such that $\delta([\nu^{1/2}\rho, \nu^{3/2}\rho]) \otimes \sigma_k$ appears in $M^*(\delta([\nu^{1/2}\rho, \nu^{3/2}\rho])) \rtimes \pi_1 \otimes \pi_2$, we have the following analysis: $\delta([\nu^{1/2}\rho, \nu^{3/2}\rho]) \otimes 1 \leq M^*(\delta([\nu^{1/2}\rho, \nu^{3/2}\rho]))$ satisfies the requirements with $\pi_1 = 1$ and $\pi_2 = \sigma_k$, the representations $\delta([\nu^{-3/2}\rho, \nu^{1/2}\rho]) \otimes 1$ and $\nu^{-1/2}\rho \times \nu^{3/2} \otimes 1$ obviously do not, the same goes for $\nu^{-1/2}\rho \otimes \nu^{3/2}\rho$. The ones to check are $\nu^{3/2}\rho \otimes \nu^{1/2}\rho$ and $1 \otimes \delta([\nu^{1/2}\rho, \nu^{3/2}\rho])$. But in these cases $\pi_1$ should have $\nu^{1/2}\rho_{k+1}$ in the cuspidal support, which is not the case. ∎

*Proof of Lemma 5.1.* We will prove the unitarizability using the same basic idea as in Lemma 3.7. In the course of the proof we use some ideas from ([20, Section 8]).

By the induction hypothesis, the representation $\widehat{\sigma_k}$ is unitarizable. We write down the Langlands' parameter of the representation $\widehat{\sigma_k}$ as follows: $\widehat{\sigma_k} = L(\delta_1\nu^{s_1}, \ldots, \delta_l\nu^{s_l}; \tau)$. The representations $\delta_i,\ i = 1, \ldots, l$ are the discrete series representations of the general linear groups; they do not have any twist of the representation $\rho = \rho_{k+1}$ in their cuspidal supports. The representation $\tau$ is a tempered representation with $\tau \hookrightarrow \delta_1' \times \cdots \times \delta_t' \rtimes \sigma$, where $\delta_i'$ are again discrete series representations without $\rho$ in their cuspidal support; the representation $\sigma$ is a discrete series representation with $\sigma_{cusp} = \sigma'$. We have $s_l \in 1/2\mathbb{Z}_+$. We have an isomorphism:

$$\nu^{3/2}\rho \times \nu^{1/2}\rho \times \delta_1\nu^{s_1} \times \cdots \times \delta_l\nu^{s_l} \rtimes \tau \cong \delta_1\nu^{s_1} \times \cdots \times \delta_l\nu^{s_l} \times \nu^{3/2}\rho \times \nu^{1/2}\rho \rtimes \tau.$$

We leave the right-hand side of the previous relation as it is if $s_l \geq 3/2$, if not, we just move $\nu^{3/2}\rho$ (isomorphically) until we get a standard representation on the right-hand side. The representation on the right-hand side is standard and has a unique irreducible quotient. There is an epimorphism from that representation to $L(\nu^{3/2}\rho, \nu^{1/2})\rho \rtimes L(\delta_1\nu^{s_1}, \ldots, \delta_l\nu^{s_l}; \tau)$, so we see that then $\widehat{\sigma_{k+1,1,2}} = L(\delta_1\nu^{s_1}, \ldots, \delta_l\nu^{s_l}, \nu^{3/2}\rho, \nu^{1/2}\rho; \tau)$. (For simplicity of notation, we assumed $s_l \geq 3/2$, but if it is not so, the following procedure also holds.) We have the following epimorphism.

$$\nu^{-1/2}\rho \times \delta_1\nu^{s_1} \times \cdots \times \delta_l\nu^{s_l} \times \nu^{3/2} \times \nu^{1/2}\rho \rtimes \tau \to \nu^{-1/2}\rho \rtimes \widehat{\sigma_{k+1,1,2}}.$$

The left-hand of the previous relation is isomorphic to $\delta_1\nu^{s_1} \times \cdots \times \delta_l\nu^{s_l} \times \nu^{3/2}\rho \times \nu^{-1/2}\rho \times \nu^{1/2}\rho \rtimes \tau$, so we have an epimorphism:

$$\delta_1\nu^{s_1} \times \cdots \times \delta_l\nu^{s_l} \times \nu^{3/2}\rho \times \nu^{-1/2}\rho \times \nu^{1/2}\rho \rtimes \tau \to \nu^{-1/2}\rho \rtimes \widehat{\sigma_{k+1,1,2}}.$$

Assume that the restriction $\phi$ of this epimorphism to the representation space $\delta_1\nu^{s_1} \times \cdots \times \delta_l\nu^{s_l} \times \nu^{3/2}\rho \times L(\nu^{1/2}\rho, \nu^{-1/2}\rho) \rtimes \tau$ is still an epimorphism. We will now prove that this is not the case.

If this were the case, we should have

$$\mu^*(\delta_1\nu^{s_1} \times \cdots \times \delta_l\nu^{s_l} \times \nu^{3/2}\rho \times L(\nu^{1/2}\rho, \nu^{-1/2}\rho) \rtimes \tau) \geq \mu^*(\nu^{-1/2}\rho \rtimes \widehat{\sigma_{k+1,1,2}}).$$

To simplify notation, we introduce $\pi = \delta_1\nu^{s_1} \times \cdots \times \delta_l\nu^{s_l}$. We then have

$$\pi \times \nu^{\frac{3}{2}}\rho \times L(\nu^{1/2}\rho, \nu^{-1/2}\rho) \rtimes \tau \cong \nu^{3/2}\rho \times L(\nu^{1/2}\rho, \nu^{-1/2}\rho) \rtimes (\pi \rtimes \tau).$$

Let $s$ be an integer such that the representation $\rho$ is a representation of $GL(s, F)$. Now we compare the Jacquet modules with respect to the maximal standard parabolic subgroup with the Levi subgroup isomorphic to $GL(3s, F) \times G_{m'}$, for the appropriate $m'$, of the representations $\nu^{3/2}\rho \times L(\nu^{1/2}\rho, \nu^{-1/2}\rho) \rtimes (\pi \rtimes \tau)$ and $\nu^{-1/2}\rho \rtimes \widehat{\sigma_{k+1,1,2}}$. We have the epimorphisms

$$\nu^{3/2}\rho \times \nu^{1/2}\rho \times \pi \rtimes \tau \to \widehat{\sigma_{k+1,1,2}} = L(\delta_1\nu^{s_1}, \ldots, \delta_l\nu^{s_l}, \nu^{3/2}\rho, \nu^{1/2}\rho; \tau)$$

and

$$L(\nu^{3/2}\rho, \nu^{1/2}\rho) \rtimes (\pi \rtimes \tau) \to L(\delta_1\nu^{s_1}, \ldots, \delta_l\nu^{s_l}, \nu^{3/2}\rho, \nu^{1/2}\rho; \tau)$$

By the Frobenius reciprocity, we have

$$\mathrm{Hom}(L(\widetilde{\nu^{3/2}\rho, \nu^{1/2}}\rho) \otimes (\pi \rtimes \tau),$$
$$r_{GL(3s,F)\times G_{m'}}(L(\delta_1\nu^{s_1}, \ldots, \delta_l\nu^{s_l}, \nu^{3/2}\rho, \nu^{1/2}\rho; \tau))) \neq 0.$$

So, the Jacquet module $r_{GL(3s,F)\times G_{m'}}(L(\delta_1\nu^{s_1},\dots,\delta_l\nu^{s_l},\nu^{3/2}\rho,\nu^{1/2}\rho;\tau))$ has an irreducible subquotient of the form $L(\nu^{-1/2}\rho,\nu^{-3/2}\rho)\otimes\pi_1$, where $\pi_1$ is an irreducible subquotient of $\pi\rtimes\tau$. From this follows that

$$\nu^{1/2}\rho\times L(\nu^{-1/2}\rho,\nu^{-3/2}\rho)\otimes\pi_1\leq r_{GL(3s,F)\times G_{m'}}(\nu^{-1/2}\rho\rtimes\widehat{\sigma_{k+1,1,2}}).$$

The representation $\nu^{1/2}\rho\times L(\nu^{-1/2}\rho,\nu^{-3/2}\rho)$ has a unique irreducible representation, $Z(\{1/2\},\{-1/2,-3/2\})$, in the Zelevinsky's notation. We will prove that the subquotient $Z(\{1/2\},\{-1/2,-3/2\})\otimes\pi_1$ does not appear in $r_{GL(3s,F)\times G_{m'}}(\nu^{3/2}\rho\times L(\nu^{1/2}\rho,\nu^{-1/2}\rho)\rtimes(\pi\rtimes\tau))$. Using the structure formula for the Jacquet modules, we get that the only subquotients of

$$r_{GL(3s,F)\times G_{m'}}(\nu^{3/2}\rho\times L(\nu^{1/2}\rho,\nu^{-1/2}\rho)\rtimes(\pi\rtimes\tau))$$

which have only twists of $\rho$ in the cuspidal support of the first factor, are of the following form:

$$\nu^{\frac{3}{2}}\rho\times L(\nu^{3/2}\rho,\nu^{1/2}\rho)\otimes\pi_1',\nu^{3/2}\rho\times\nu^{-1/2}\rho\times\nu^{-1/2}\rho\otimes\pi_1',$$
$$\nu^{-3/2}\rho\times L(\nu^{3/2}\rho,\nu^{1/2}\rho)\otimes\pi_1',\ \nu^{-3/2}\rho\times\nu^{-1/2}\rho\times\nu^{-1/2}\rho\otimes\pi_1'.$$

None of them has $Z(\{1/2\},\{-1/2,-3/2\})\otimes\pi_1$ as a subquotient.

From this we conclude that there is a non-zero intertwining

$$\psi:\pi\times\nu^{3/2}\rho\times\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\rtimes\tau\to\nu^{-1/2}\rho\rtimes\widehat{\sigma_{k+1,1,2}}/\mathrm{Im}\phi.$$

Since $\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])$ does not appear in the tempered support of $\tau$, and $\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\rtimes\sigma$ is reducible, by [14, Lemma 1.2 (ii)], we know that the representation $\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\rtimes\tau$ is sum of two non-equivalent tempered representations, say $T_1$ and $T_2$.

Using formulas for the Jacquet modules, we get that

$$\begin{aligned}r_{GL(2s,F)\times G_{m''}}(T_1+T_2)=&2\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\otimes\tau+\nu^{1/2}\rho\times\nu^{1/2}\rho\otimes\tau\\&+\sum 2\nu^{1/2}\rho\times\pi_1'\otimes\nu^{-1/2}\rho\rtimes\pi_2'\\&+\pi_1''\otimes\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\rtimes\pi_2''.\end{aligned}$$

The sum is over all appropriate irreducible $\pi_1'\otimes\pi_2',\ \pi_1''\otimes\pi_2''\in\mu^*(\tau)$. Since $\mathrm{Hom}(T_i,\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\rtimes\tau)\neq 0$, we get, say

$$r_{GL(2s,F)\times G_{m''}}(T_1)\geq\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\otimes\tau+\nu^{1/2}\rho\times\nu^{1/2}\rho\otimes\tau,$$
$$r_{GL(2s,F)\times G_{m''}}(T_2)\geq\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\otimes\tau.$$

If we assume that

$$\psi|_{\pi\times\nu^{3/2}\rho\rtimes T_1}:\pi\times\nu^{3/2}\rho\rtimes T_1\to\nu^{-1/2}\rho\rtimes\widehat{\sigma_{k+1,1,2}}/\mathrm{Im}\phi$$

is non-zero, we will get that $L(\pi,\nu^{3/2}\rho;T_1)\leq\nu^{-1/2}\rho\rtimes\widehat{\sigma_{k+1,1,2}}$. But then, using the Frobenius reciprocity, and fact that $\nu^{1/2}\rho\times\nu^{1/2}\rho\otimes\tau\leq\mu^*(T_1)$, we get that

$$\lambda\otimes\nu^{-3/2}\rho\otimes\nu^{1/2}\rho\times\nu^{1/2}\rho\otimes\tau\leq\mu^*(\nu^{-1/2}\rho\rtimes\widehat{\sigma_{k+1,1,2}}),$$

for some irreducible subquotient $\lambda$ of $\widetilde{\pi}$.

We will prove that this cannot hold, moreover we will prove that $\lambda\otimes\nu^{-3/2}\rho\otimes\nu^{1/2}\rho\times\nu^{1/2}\rho\otimes\tau$ is not a subquotient of the appropriate Jacquet module of $\pi\times\ \nu^{-1/2}\rho\rtimes L(\nu^{3/2}\rho,\nu^{1/2}\rho;\tau)$.

We observe that $\tau\leq\alpha\rtimes\sigma'$, where $\alpha$ is an irreducible representation of an appropriate general linear group, without twists of $\rho$ in the cuspidal support. Then it is not difficult to see that

$$L(\nu^{3/2}\rho,\nu^{1/2}\rho;\tau)\leq\alpha\rtimes L(\nu^{3/2}\rho,\nu^{1/2}\rho;\sigma').$$

So, we will actually prove that $\lambda\otimes\nu^{-3/2}\rho\otimes\nu^{1/2}\rho\times\nu^{1/2}\rho\otimes\tau$ is not a subquotient of $\mu^*(\pi\times\ \nu^{-1/2}\rho\times\alpha\rtimes L(\nu^{3/2}\rho,\nu^{1/2}\rho;\sigma'))$. First, we will analyze all the possible subquotients of $\mu^*(\pi\times\ \nu^{-1/2}\rho\times\alpha\rtimes L(\nu^{3/2}\rho,\nu^{1/2}\rho;\sigma'))$ of the form $\lambda\otimes\zeta$. We calculate

$$M^*(\pi\times\alpha)\times M^*(\nu^{-1/2}\rho)\rtimes\mu^*(L(\nu^{3/2}\rho,\nu^{1/2}\rho;\sigma')).$$

We further have

$$M^*(\nu^{-1/2}\rho)=\nu^{-1/2}\rho\otimes 1+\nu^{1/2}\rho\otimes 1+1\otimes\nu^{-1/2}\rho,$$

and

$$\begin{aligned}&\mu^*(L(\nu^{3/2}\rho,\nu^{1/2}\rho;\sigma'))\\&=1\otimes L(\nu^{3/2}\rho,\nu^{1/2}\rho;\sigma')+\nu^{-3/2}\rho\otimes L(\nu^{1/2}\rho;\sigma')+L(\nu^{-1/2}\rho,\nu^{-3/2}\rho)\otimes\sigma'.\end{aligned}$$

If we take $\pi_1\otimes\pi_2\in M^*(\pi\times\alpha)$, $\pi_1'\otimes\pi_2'\in M^*(\nu^{-1/2}\rho)$ and $\pi_1''\otimes\pi_2''\in\mu^*(L(\nu^{3/2}\rho,\nu^{1/2}\rho;\sigma'))$, then

$$\lambda\otimes\zeta\leq\pi_1\times\pi_1'\times\pi_1''\otimes\pi_2\times\pi_2'\times\pi_2''.$$

Since $\lambda$ does not have twists of $\rho$ in the cuspidal support, we get $\pi_1'\otimes\pi_2'=1\otimes\nu^{-1/2}\rho$ and $\pi_1''\otimes\pi_2''=1\otimes L(\nu^{3/2}\rho,\nu^{1/2}\rho;\sigma')$, so

$$\lambda\otimes\zeta\leq\pi_1\otimes\pi_2\times\nu^{-1/2}\rho\rtimes L(\nu^{3/2}\rho,\nu^{1/2}\rho;\sigma').$$

Now we want to examine the appropriate Jacquet module of $\zeta$, such that in the second factor we do not have twists of $\rho$ in the cuspidal support. It turns out that the appropriate subquotients are of the form $\zeta_1 \times L(\nu^{-1/2}\rho, \nu^{-3/2}\rho) \times \nu^{\pm 1/2}\rho \otimes \zeta_2 \rtimes \sigma'$, where $\zeta_1 \otimes \zeta_2 \in \mu^*(\pi_2)$. We want $\tau$ to be a subquotient of $\zeta_2 \rtimes \sigma'$, and when we compare the dimensions of the groups involved, we see that $\zeta_1 = 1$, so we actually want to examine if $\nu^{-3/2}\rho \otimes \nu^{1/2}\rho \times \nu^{1/2}\rho \leq m^*(L(\nu^{-1/2}\rho, \nu^{-3/2}\rho) \times \nu^{\pm 1/2}\rho)$. This cannot hold, because, we do not have $\nu^{1/2}\rho$ two times in the cuspidal support on the right-hand side, as we have for the left-hand side.

From this we conclude that $L(\pi, \nu^{3/2}\rho; T_2) \leq \nu^{-1/2}\rho \rtimes \widehat{\sigma_{k+1,1,2}}$. The other subquotient of $\nu^{-1/2}\rho \rtimes \widehat{\sigma_{k+1,1,2}}$ already appears as a subquotient of $\widehat{\delta([\nu^{-1/2}\rho, \nu^{3/2}\rho])} \rtimes \widehat{\sigma_k}$, so we know that it is unitarizable. We want to prove the unitarizability of $L(\pi, \nu^{3/2}\rho; T_2)$. We use the same idea as in Lemma 3.7, but now we have a bit more elaborated proof. By [14, Lemmas 6.1 and 6.2], and since $\mathrm{Jord}_\rho(\sigma) = \mathrm{Jord}_\rho(\sigma')$, $\delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \rtimes \sigma = \tau_1 \oplus \tau_2$ for some tempered representations $\tau_i$, $i = 1, 2$. Using the factorization of the long intertwining operator, we see that $\nu^{1/2}\rho \rtimes T_i$ reduces if and only if $\nu^{1/2}\rho \rtimes \tau_i$ reduces (where $T_i \hookrightarrow \delta_1' \times \cdots \times \delta_t' \rtimes \tau_i$). Here, $\tau_2$ is a representation which does not have $\nu^{1/2}\rho \times \nu^{1/2}\rho \otimes \sigma$ in its Jacquet module. Analogously to what we have done, we prove that the representation $\nu^{1/2}\rho \rtimes \tau_2$ does not reduce; otherwise, it would have $\delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \rtimes \sigma_0$ as a subquotient, and again, by comparing Jacquet modules, we see that it is not possible. Here $\sigma_0$ is a representation for which $\mathrm{Jord}(\sigma_0) = \mathrm{Jord}(\sigma) \cup \{(2, \rho)\}$. In this way, we have proved that the representation $\nu^{1/2}\rho \rtimes T_2$ is irreducible. We conclude

$$\nu^{1/2}\rho \times \widetilde{\pi} \rtimes T_2 \cong \widetilde{\pi} \times \nu^{1/2}\rho \rtimes T_2 \cong \widetilde{\pi} \times \nu^{-1/2}\rho \rtimes T_2 \cong \nu^{-1/2}\rho \times \widetilde{\pi} \rtimes T_2.$$

Examining the restriction of the corresponding intertwining operator, we see that

$$\nu^{1/2}\rho \rtimes L(\pi; T_2) \cong \nu^{-1/2}\rho \rtimes L(\pi; T_2).$$

Let $\alpha$ denote an irreducible subrepresentation of $\nu^{1/2}\rho \rtimes L(\pi; T_2)$. Then $\widetilde{\alpha}$ is a quotient of $\nu^{-1/2}\rho \rtimes \widetilde{L(\pi; T_2)}$. The representation $\delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \times \pi \rtimes \tau$ has exactly two irreducible quotients, $L(\pi; T_i)$, $i = 1, 2$, both contained in $\delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \rtimes L(\pi; \tau)$; we see that both of them are unitarizable ( $L(\pi; \tau) = \widehat{\sigma_k}$), so $\widetilde{\alpha} \cong \overline{\alpha}$, and $\widetilde{L(\pi; T_2)} = \overline{L(\pi; T_2)}$. Since $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\overline{\sigma'})$, we get $\overline{\alpha} \hookrightarrow \nu^{1/2}\rho \rtimes L(\pi; \overline{T_2}) \cong \nu^{-1/2}\rho \rtimes \overline{L(\pi; T_2)}$, and so $\overline{\alpha} = L(\pi, \nu^{1/2}\rho; \overline{T_2})$, and

$\alpha = L(\pi, \nu^{1/2}\rho; T_2)$. The representation $\nu^{1/2}\rho \rtimes L(\pi; T_2)$ is irreducible. All the subquotients of the representation $\nu^{3/2}\rho \rtimes L(\pi; T_2)$ are unitarizable, and so is $L(\nu^{3/2}\rho, \pi; T_2)$. ∎

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